A Posteriori Error Estimates for the Biharmonic Equation

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Overview

The Biharmonic Equation

$$\begin{aligned} \Delta^2 u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \Gamma \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \Gamma \end{aligned}$$

- Models the vertical displacement u of the mid-surface of a thin clamped plate under the influence of a vertical load f.
- Models the vorticity of a two-dimensional Stokes flow.

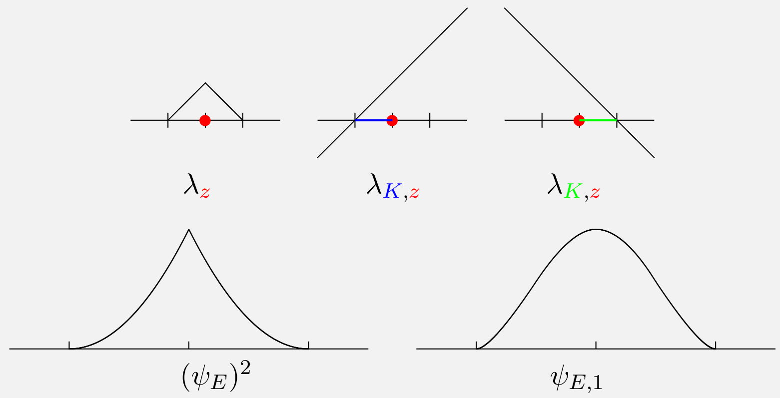
Variational Formulation and Discretization

- Find $u \in H_0^2(\Omega)$ such that for all $v \in H_0^2(\Omega)$
$$\int_\Omega \Delta u \Delta v = \int_\Omega f v.$$
- Find $u_\mathcal{T} \in X_\mathcal{T} \subset H_0^2(\Omega)$ such that for all $v_\mathcal{T} \in X_\mathcal{T}$
$$\int_\Omega \Delta u_\mathcal{T} \Delta v_\mathcal{T} = \int_\Omega f v_\mathcal{T}.$$
- Requires C^1-elements.

Proof of A Posteriori Error Estimates

- Upper bound is standard.
  - Conformity $X_{\mathcal{T}} \subset H_0^2(\Omega)$ implies Galerkin orthogonality.
  - Integration by parts twice element-wise yields L^2-representation.
  - Standard approximation properties of nodal interpolation prove upper bound.
- Lower bound requires C^1-cut-off functions.
  - $\prod_{z\in\mathcal{N}_K} \lambda_z^2$ controls the element residual $f_{\mathcal{T}} - \Delta^2 u_{\mathcal{T}}$.
  - $\psi_{E,1} = \prod_{K\subset\omega_E} \prod_{z\in\mathcal{N}_K} \lambda_{K,z}^2 \chi_{\omega_E}$ controls the edge residual $\mathbb{J}_E(\mathbf{n}_E \cdot \nabla\Delta u_{\mathcal{T}})$.
  - $\psi_{E,1} \left(\frac{|K_2|}{|E|} \sum_{z\in\mathcal{N}_{K_2}\setminus\mathcal{N}_E} \lambda_{K_2,z} - \frac{|K_1|}{|E|} \sum_{z\in\mathcal{N}_{K_1}\setminus\mathcal{N}_E} \lambda_{K_2,z} \right)$ controls the edge residual $\mathbb{J}_E(\Delta u_{\mathcal{T}})$.

Smooth Cut-off Functions



A Posteriori Error Estimates

$$
\begin{aligned}
\|u - u_{\mathcal{T}}\|_2 \approx \Bigg\{ & \sum_{K\in\mathcal{T}} h_K^4 \left\| f_{\mathcal{T}} - \Delta^2 u_{\mathcal{T}} \right\|_K^2 \\
& + \sum_{E\in\mathcal{E}_\Omega} h_E \left\| \mathbb{J}_E(\Delta u_{\mathcal{T}}) \right\|_E^2 \\
& + \sum_{E\in\mathcal{E}_\Omega} h_E^3 \left\| \mathbb{J}_E(\mathbf{n}_E \cdot \nabla\Delta u_{\mathcal{T}}) \right\|_E^2 \\
& + \sum_{K\in\mathcal{T}} h_K^4 \left\| f - f_{\mathcal{T}} \right\|_K^2 \Bigg\}^{\frac{1}{2}}
\end{aligned}
$$

Mixed Variational Problem and Discretization

- Find $\varphi \in H^1(\Omega)$ and $u \in H_0^1(\Omega)$ such that for all $\psi \in H^1(\Omega)$ and $v \in H_0^1(\Omega)$
$$
\begin{aligned}
\int_\Omega \varphi\psi + \int_\Omega \nabla\psi \cdot \nabla u &= 0 \\
\int_\Omega \nabla v \cdot \nabla\varphi &= -\int_\Omega f v
\end{aligned}
$$
- Find $\varphi_{\mathcal{T}} \in V_{\mathcal{T}} \subset H^1(\Omega)$ and $u_{\mathcal{T}} \in W_{\mathcal{T}} = V_{\mathcal{T}} \cap H_0^1(\Omega)$ such that for all $\psi_{\mathcal{T}} \in V_{\mathcal{T}}$ and $v_{\mathcal{T}} \in W_{\mathcal{T}}$
$$
\begin{aligned}
\int_\Omega \varphi_{\mathcal{T}}\psi_{\mathcal{T}} + \int_\Omega \nabla\psi_{\mathcal{T}} \cdot \nabla u_{\mathcal{T}} &= 0 \\
\int_\Omega \nabla v_{\mathcal{T}} \cdot \nabla\varphi_{\mathcal{T}} &= -\int_\Omega f v_{\mathcal{T}}
\end{aligned}
$$
- The inf-sup condition is violated.

Residuals

$$\begin{aligned}\langle R_1\,,\,\psi\rangle &= \int_\Omega \varphi_\mathcal{T}\psi + \int_\Omega \nabla u_\mathcal{T}\cdot\nabla\psi\\ &= \sum_{K\in\mathcal{T}}\int_K(\varphi_\mathcal{T}-\Delta u_\mathcal{T})\,\psi + \sum_{E\in\mathcal{E}}\int_E \mathbb{J}_E(\mathbf{n}_E\cdot\nabla u_\mathcal{T})\psi\\ \langle R_2\,,\,v\rangle &= \int_\Omega \nabla\varphi_\mathcal{T}\cdot\nabla v + \int_\Omega fv\\ &= \sum_{K\in\mathcal{T}}\int_K(f-\Delta\varphi_\mathcal{T})\,v + \sum_{E\in\mathcal{E}_\Omega}\int_E \mathbb{J}_E(\mathbf{n}_E\cdot\nabla\varphi_\mathcal{T})v\end{aligned}$$

Lower Bounds for R_1

- Test-functions $\psi=\psi_K(\varphi_\mathcal{T}-\Delta u_\mathcal{T})$, $v=0$ yield
$$\begin{aligned}\|\varphi_\mathcal{T}-\Delta u_\mathcal{T}\|_K^2 \lesssim\ & h_K^{-1}\|\nabla(u-u_\mathcal{T})\|_K\|\varphi_\mathcal{T}-\Delta u_\mathcal{T}\|_K\\ &+\|\varphi-\varphi_\mathcal{T}\|_K\|\varphi_\mathcal{T}-\Delta u_\mathcal{T}\|_K\end{aligned}$$
- Test-functions $\psi=\psi_E\mathbb{J}_E(\mathbf{n}_E\cdot\nabla u_\mathcal{T})$, $v=0$ give
$$h_E^{\frac12}\|\mathbb{J}_E(\mathbf{n}_E\cdot\nabla\varphi_\mathcal{T})\|_E \lesssim \|\nabla(u-u_\mathcal{T})\|_{\omega_E}+h_E\|\varphi-\varphi_\mathcal{T}\|_{\omega_E}$$
- Mesh-dependent norm $\left\{\|\nabla v\|^2+\sum_K h_K^2\|\psi\|_K^2\right\}^{\frac12}$ may be a candidate for measuring the error.
- Causes problems with R_2 which is linked to $\nabla(\varphi-\varphi_\mathcal{T})$.

Lower Bounds for R_2

- Test-functions $\psi=0$, $v=v_K=\psi_{K,1}(f_\mathcal{T}-\Delta u_\mathcal{T})$ and integration by parts for the $(\varphi-\varphi_\mathcal{T})$-term yield
$$h_K^3\|f_\mathcal{T}-\Delta u_\mathcal{T}\|_K \lesssim h_K\|\varphi-\varphi_\mathcal{T}\|_K+h_K^3\|f-f_\mathcal{T}\|_K$$
- Test-functions $\psi=0$, $v=\psi_{E,1}\mathbb{J}_E(\mathbf{n}_E\cdot\nabla\varphi_\mathcal{T})$ and integration by parts for the $(\varphi-\varphi_\mathcal{T})$-term give
$$h_E^{\frac52}\|\mathbb{J}_E(\mathbf{n}_E\cdot\nabla\varphi_\mathcal{T})\|_E \lesssim h_E\|\varphi-\varphi_\mathcal{T}\|_{\omega_E}+h_E^3\|f-f_\mathcal{T}\|_{\omega_E}$$
- Error estimator
$$\left\{\sum_{K\in\mathcal{T}}h_K^2\|\varphi_\mathcal{T}-\Delta u_\mathcal{T}\|_K^2+\sum_{E\in\mathcal{E}}h_E\|\mathbb{J}_E(\mathbf{n}_E\cdot\nabla u_\mathcal{T})\|_E^2\right.$$
$$\left.+\sum_{K\in\mathcal{T}}h_K^6\|f_\mathcal{T}-\Delta\varphi_\mathcal{T}\|_K^2+\sum_{E\in\mathcal{E}_\Omega}h_E^5\|\mathbb{J}_E(\mathbf{n}_E\cdot\nabla\varphi_\mathcal{T})\|_E^2\right\}^{\frac12}$$

Upper Bound for $\|\nabla(u-u_\mathcal{T})\|$

- Use a duality argument.
- Choose $g\in H^{-1}(\Omega)$ such that $\|g\|_{-1}=1$ and $\langle g\,,\,u-u_\mathcal{T}\rangle=\|\nabla(u-u_\mathcal{T})\|$.
- Denote by u_g the solution of a biharmonic equation with right-hand side g; set $\varphi_g=\Delta u_g$.
- Insert $u-u_\mathcal{T}$ as test-function in the mixed formulation of the auxiliary biharmonic equation.
- Use the Galerkin orthogonality and standard approximation properties of the nodal interpolation to bound $\langle g\,,\,u-u_\mathcal{T}\rangle$ in terms of the error estimator and $\|\varphi_g\|_1+\|u_g\|_3$.
- Use the regularity result $\|\varphi_g\|_1+\|u_g\|_3\lesssim\|g\|_{-1}$ for convex domains.

Upper Bound for $\left\{\sum_K h_K^2 \|\varphi - \varphi_{\mathcal{T}}\|_K^2\right\}^{\frac{1}{2}}$

- Introduce a mesh-function $h \in W^{1,\infty}(\Omega)$ and assume that $h_K \approx h_{|K}$ for every element K.
- Insert $\psi = h^2(\varphi - \varphi_{\mathcal{T}})$, $v = -h^2(u - u_{\mathcal{T}})$ in the error equation to obtain

$$\begin{aligned}\|h(\varphi - \varphi_{\mathcal{T}})\|^2 = \;& \left\langle R_1\,,\, h^2(\varphi - \varphi_{\mathcal{T}})\right\rangle \\ & - \left\langle R_2\,,\, h^2(u - u_{\mathcal{T}})\right\rangle \\ & -2\int_\Omega h(\varphi - \varphi_{\mathcal{T}})\nabla h \cdot \nabla(u - u_{\mathcal{T}}) \\ & +2\int_\Omega h(u - u_{\mathcal{T}})\nabla(\varphi - \varphi_{\mathcal{T}}) \cdot \nabla h.\end{aligned}$$

- Bound the right-hand side using Galerkin orthogonality, interpolation error estimates, and inverse estimates.

A Posteriori Error Estimates

$$\begin{aligned}&\left\{\sum_K h_K^2 \|\varphi - \varphi_{\mathcal{T}}\|_K^2\right\}^{\frac{1}{2}} + \|\nabla(u - u_{\mathcal{T}})\| \\ &\approx \left\{\sum_{K\in\mathcal{T}} h_K^2 \|\varphi_{\mathcal{T}} - \Delta u_{\mathcal{T}}\|_K^2 + \sum_{E\in\mathcal{E}} h_E \|\mathbb{J}_E(\mathbf{n}_E \cdot \nabla u_{\mathcal{T}})\|_E^2 \right.\\ &\qquad + \sum_{K\in\mathcal{T}} h_K^6 \|f_{\mathcal{T}} - \Delta\varphi_{\mathcal{T}}\|_K^2 + \sum_{E\in\mathcal{E}_\Omega} h_E^5 \|\mathbb{J}_E(\mathbf{n}_E \cdot \nabla\varphi_{\mathcal{T}})\|_E^2 \\ &\qquad \left. + \sum_{K\in\mathcal{T}} h_K^6 \|f - f_{\mathcal{T}}\|_K^2 \right\}^{\frac{1}{2}} + \max_{K\in\mathcal{T}} h_K^2 \|f\|_{-1}\end{aligned}$$

Discretization

Find $u_{\mathcal{T}} \in S_0^{2,0}(\mathcal{T})$ (continuous, piecewise quadratic, vanishing on Γ) such that for all $v_{\mathcal{T}} \in S_0^{2,0}(\mathcal{T})$

$$\begin{aligned}\int_\Omega f v_{\mathcal{T}} = & \sum_{K\in\mathcal{T}} \int_K D^2 u_{\mathcal{T}} : D^2 v_{\mathcal{T}} \\ & + \sum_{E\in\mathcal{E}} \int_E \mathbb{A}_E(\mathbf{n}_E \cdot D^2 u_{\mathcal{T}} \mathbf{n}_E) \mathbb{J}_E(\mathbf{n}_E \cdot \nabla v_{\mathcal{T}}) \\ & + \sum_{E\in\mathcal{E}} \int_E \mathbb{J}_E(\mathbf{n}_E \cdot \nabla u_{\mathcal{T}}) \mathbb{A}_E(\mathbf{n}_E \cdot D^2 v_{\mathcal{T}} \mathbf{n}_E) \\ & + \sigma \sum_{E\in\mathcal{E}} h_E^{-1} \int_E \mathbb{J}_E(\mathbf{n}_E \cdot \nabla u_{\mathcal{T}}) \mathbb{J}_E(\mathbf{n}_E \cdot \nabla v_{\mathcal{T}}).\end{aligned}$$

Mesh-dependent Norm

- Mesh-dependent semi-norm

$$|v_{\mathcal{T}}|_{\mathcal{T}} = \left\{\sum_{K\in\mathcal{T}} \|v_{\mathcal{T}}\|_{2;K}^2\right\}^{\frac{1}{2}}$$

- Mesh-dependent norm

$$|||v_{\mathcal{T}}|||_{\mathcal{T}} = \left\{|v_{\mathcal{T}}|_{\mathcal{T}}^2 + \sigma \sum_{E\in\mathcal{E}} h_E^{-1} \|\mathbb{J}_E(\mathbf{n}_E \cdot \nabla v_{\mathcal{T}})\|_E^2\right\}^{\frac{1}{2}}$$

- $u \in H_0^2(\Omega)$ implies

$$|||u - u_{\mathcal{T}}|||_{\mathcal{T}} = \left\{|u - u_{\mathcal{T}}|_{\mathcal{T}}^2 + \sigma \sum_{E\in\mathcal{E}} h_E^{-1} \|\mathbb{J}_E(\mathbf{n}_E \cdot \nabla u_{\mathcal{T}})\|_E^2\right\}^{\frac{1}{2}}.$$

Lifting to a Conforming Subspace of $H_0^2(\Omega)$

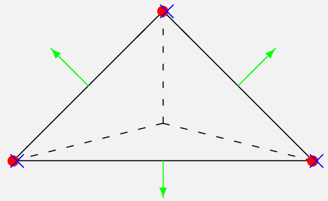
- Compare $u - u_{\mathcal{T}}$ with $u - E_{\mathcal{T}} u_{\mathcal{T}}$ where $E_{\mathcal{T}} u_{\mathcal{T}}$ is a lifting of $u_{\mathcal{T}}$ to the Hsieh-Clough-Tougher subspace of $H_0^2(\Omega)$.
- $E_{\mathcal{T}} u_{\mathcal{T}}$ is defined by

$$
\begin{aligned}
E_{\mathcal{T}} u_{\mathcal{T}}(z) &= u_{\mathcal{T}}(z), \\
\nabla(E_{\mathcal{T}} u_{\mathcal{T}})(z) &= \frac{1}{\#\{K \subset \omega_z\}} \sum_{K \subset \omega_z} \nabla u_{\mathcal{T}}|_K(z), \\
\mathbf{n}_E \cdot \nabla(E_{\mathcal{T}} u_{\mathcal{T}})(z_E) &= \mathbb{A}_E(\mathbf{n}_E \cdot u_{\mathcal{T}})(z_E).
\end{aligned}
$$

- $|u - u_{\mathcal{T}}|_{\mathcal{T}} \le \left\| D^2(u - E_{\mathcal{T}} u_{\mathcal{T}}) \right\| + |E_{\mathcal{T}} u_{\mathcal{T}} - u_{\mathcal{T}}|_{\mathcal{T}}$

Hsieh-Clough-Tougher Element

$$\left\{ v \in C^1(K) : v_{|K_i} \in \mathbb{P}_3 \right\}$$



function values, first order derivatives, normal derivatives

Estimation of $|E_{\mathcal{T}} u_{\mathcal{T}} - u_{\mathcal{T}}|_{\mathcal{T}}$

- A local inverse estimate yields

$$|u_{\mathcal{T}} - E_{\mathcal{T}} u_{\mathcal{T}}|_{\mathcal{T}} \lesssim \left\{ \sum_{K \in \mathcal{T}} h_K^{-4} \left\| u_{\mathcal{T}} - E_{\mathcal{T}} u_{\mathcal{T}} \right\|_K^2 \right\}^{\frac{1}{2}}.$$

- A scaling argument gives

$$
\begin{aligned}
h_K^{-4} \left\| u_{\mathcal{T}} - E_{\mathcal{T}} u_{\mathcal{T}} \right\|_K^2 \lesssim &\sum_{z \in \mathcal{N}_K} |\nabla(u_{\mathcal{T}} - E_{\mathcal{T}} u_{\mathcal{T}})(z)|^2 \\
&+ \left\{ \sum_{E \in \mathcal{E}_K} h_E^{-1} \left\| \mathbb{J}_E(\mathbf{n}_E \cdot \nabla u_{\mathcal{T}}) \right\|_E^2 \right\}^{\frac{1}{2}}.
\end{aligned}
$$

- The first term on the right-hand side can be controlled by the second one (same argument as for the ZZ-estimator).

Estimation of $\left\| D^2(u - E_{\mathcal{T}} u_{\mathcal{T}}) \right\|$

- Since $u - E_{\mathcal{T}} u_{\mathcal{T}} \in H_0^2(\Omega)$ there is a $v \in H_0^2(\Omega)$ with $\|v\|_2 = 1$ and $\left\| D^2(u - E_{\mathcal{T}} u_{\mathcal{T}}) \right\| = \int_\Omega D^2(u - E_{\mathcal{T}} u_{\mathcal{T}}) : D^2 v$.
- The term on the right-hand side can be rewritten and then estimated with the help of interpolation error estimates, inverse estimates, and trace inequalities.

$$\int_\Omega D^2(u - E_\mathcal{T} u_\mathcal{T}) : D^2 v$$

$$\begin{aligned}
&= \int_\Omega f(v - i_\mathcal{T} v) - \sum_{K\in\mathcal{T}} \int_K D^2(E_\mathcal{T} u_\mathcal{T} - u_\mathcal{T}) : D^2 v \\
&\quad - \sum_{K\in\mathcal{T}} \int_K D^2 u_\mathcal{T} : D^2(v - i_\mathcal{T} v) \\
&\quad + \sum_{E\in\mathcal{E}} \int_E \mathbb{A}_E(\mathbf{n}_E \cdot D^2 u_\mathcal{T} \mathbf{n}_E) \mathbb{J}_E(\mathbf{n}_E \cdot \nabla i_\mathcal{T} v) \\
&\quad + \sum_{E\in\mathcal{E}} \int_E \mathbb{J}_E(\mathbf{n}_E \cdot \nabla u_\mathcal{T}) \mathbb{A}_E(\mathbf{n}_E \cdot D^2(i_\mathcal{T} v)\mathbf{n}_E) \\
&\quad + \sigma \sum_{E\in\mathcal{E}} h_E^{-1} \int_E \mathbb{J}_E(\mathbf{n}_E \cdot \nabla u_\mathcal{T}) \mathbb{J}_E(\mathbf{n}_E \cdot \nabla i_\mathcal{T} v).
\end{aligned}$$

A Posteriori Error Estimates

$$\begin{aligned}
&\left\{ \sum_{K\in\mathcal{T}} \|u - u_\mathcal{T}\|_{2;K}^2 \right\}^{\frac{1}{2}} \\
&\approx \left\{ \sum_{K\in\mathcal{T}} h_K^4 \left\| \Delta^2 u_\mathcal{T} - f_\mathcal{T} \right\|_K^2 \right. \\
&\qquad + \sigma^2 \sum_{E\in\mathcal{E}} h_E^{-1} \|\mathbb{J}_E(\mathbf{n}_E \cdot \nabla u_\mathcal{T})\|_E^2 \\
&\qquad + \sum_{E\in\mathcal{E}_\Omega} h_E \|\mathbb{J}_E(\Delta u_\mathcal{T})\|_E^2 \\
&\qquad \left. + \sum_{K\in\mathcal{T}} h_K^4 \|f - f_\mathcal{T}\|_K^2 \right\}^{\frac{1}{2}}
\end{aligned}$$

Summary

	error	sq. estimator	sq. oscillation
conf.	$\|u - u_\mathcal{T}\|_2$	$h_K^4 \left\| f_\mathcal{T} - \Delta^2 u_\mathcal{T} \right\|_K^2$ $+h_{\partial K} \|\mathbb{J}_E(\Delta u_\mathcal{T})\|_{\partial K}^2$ $+h_{\partial K}^3 \|\mathbb{J}_E(\mathbf{n}_E \cdot \nabla\Delta u_\mathcal{T})\|_{\partial K}^2$	$h_K^4 \|f - f_\mathcal{T}\|_K^2$
mixed	$\|h(\varphi - \varphi_\mathcal{T})\|$ $+ \|\nabla(u - u_\mathcal{T})\|$	$h_K^2 \|\varphi_\mathcal{T} - \Delta u_\mathcal{T}\|_K^2$ $+h_{\partial K} \|\mathbb{J}_E(\mathbf{n}_E \cdot \nabla u_\mathcal{T})\|_{\partial K}^2$ $+h_K^6 \|f_\mathcal{T} - \Delta\varphi_\mathcal{T}\|_K^2$ $+h_{\partial K}^5 \|\mathbb{J}_E(\mathbf{n}_E \cdot \nabla\varphi_\mathcal{T})\|_{\partial K}^2$	$h_K^6 \|f - f_\mathcal{T}\|_K^2$ $+h^2 \|f\|_{-1}$
nonconf.	$\|u - u_\mathcal{T}\|_{2;\mathcal{T}}$	$h_K^4 \left\| f_\mathcal{T} - \Delta^2 u_\mathcal{T} \right\|_K^2$ $+h_{\partial K} \|\mathbb{J}_E(\Delta u_\mathcal{T})\|_{\partial K}^2$ $+\sigma^2 h_{\partial K}^{-1} \|\mathbb{J}_E(\mathbf{n}_E \cdot \nabla u_\mathcal{T})\|_{\partial K}^2$	$h_K^4 \|f - f_\mathcal{T}\|_K^2$

References

- A. Charbonneau, K. Dossou, and R. Pierre
  A residual-based a posteriori error estimator for the Ciarlet-Raviart formulation of the first biharmonic problem
  Numer. Meth. PDE **13** (1997), no. 1, 93–111
- S. C. Brenner, T. Gudi, and L.-Y. Sung
  An a posteriori error estimator for a quadratic C^0-interior penalty method for the biharmonic problem
  IMA J. Numer. Anal. **30** (2010), no. 3, 777–798
- Handout of this talk
  www.rub.de/num1
- A Posteriori Error Estimation Techniques for Finite Element Methods
  Oxford University Press, 2013