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A stabilizing control scheme for linear systems on controlled invariant sets

Moritz Schulze Darup[†] *and* Martin Mönnigmann[†]

Abstract

We present a new stabilizing control scheme for linear discrete-time systems with input and state constraints. Essentially, we seek a controller that is able to steer all initial states within a controlled invariant set towards the origin without violating the constraints. The control law builds on a predictive control scheme. We show that a prediction horizon of n steps, where n denotes the dimension of the system, is sufficient to solve the described control task. The proposed controller is related to but different from established feedback laws associated with λ-contractive sets. In fact, the new control scheme successfully stabilizes systems, where classical λ-contractive control laws fail.

Keywords. linear discrete-time systems, input and state constraints, asymptotic stability, control laws

1 Introduction

The design of stabilizing feedback laws is an important task in control theory. For linear unconstrained systems

$$x(k+1) = A\,x(k) + B\,u(k), \qquad x(0) = x_0, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, stabilizing control laws can, for example, be systematically designed using linear quadratic regulation (LQR) [12] or pole-placement [15]. If applicable, both procedures provide stabilizing controllers that steer every initial state $x_0 \in \mathbb{R}^n$ towards the origin. Here, we consider linear systems with input and state constraints of the form

$$u(k) \in \mathcal{U} \subset \mathbb{R}^m, \qquad x(k) \in \mathcal{X} \subset \mathbb{R}^n, \qquad \forall\, k \in \mathbb{N}, \tag{2}$$

[†] M. Schulze Darup and M. Mönnigmann are with Automatic Control and Systems Theory, Department of Mechanical Engineering, Ruhr-Universität Bochum, 44801 Bochum, Germany. E-mail: moritz.schulzedarup@rub.de.

where $\mathcal{U}$ and $\mathcal{X}$ are assumed to be convex and compact sets with the origin in their interiors.

There exist various stabilizing control schemes that apply to the constrained case. Probably, the most popular approaches are minimum-time optimal control (see, e.g., [13]), constrained LQR (see, e.g., [17]) and model predictive control (MPC) (see, e.g., [2, 14]). All these controllers are capable of stabilizing every stabilizable state $x_0 \in \mathcal{X}$. However, all aforementioned approaches are also computationally demanding.

In some applications, we are not primarily interested in stabilizing all stabilizable states $x_0 \in \mathcal{X}$ but in stabilizing states x_0 in a given set $\mathcal{C} \subseteq \mathcal{X}$ using simple control schemes (see, e.g. [4, 8]). This task can easily be solved, if the set $\mathcal{C}$ is λ-contractive (see Def. 3) with $\lambda < 1$. In this case, stabilizing control laws can be designed according to the procedures presented in [4]. However, for the case $\lambda = 1$, i.e., if $\mathcal{C}$ is "only" controlled invariant, the control schemes in [4] may fail to stabilize (1) (see, e.g., the example in Sect. 3.1).

In this paper, we present a new control scheme that is capable of stabilizing linear constrained systems on controlled invariant sets. Before introducing the new controller, we have to stress that, given a controlled invariant set $\mathcal{C}$, one could always compute a λ-contractive subset of $\mathcal{C}$ (with $\lambda < 1$) according to the procedure stated in [3] and subsequently design a λ-contractive controller on this subset. However, the computation is demanding and the description of the λ-contractive subset (and consequently the associated controller) may be unnecessarily complex compared to the original controlled invariant set $\mathcal{C}$ (cp., e.g., $\mathcal{C}$ to the four λ-contractive polytopes in Fig. 2).

The approach presented here does not require to construct special subsets of $\mathcal{C}$, but it provides a stabilizing controller on any subset $\mu\mathcal{C} = \{\, \mu\, x \,|\, x \in \mathcal{C}\,\}$, $\mu \in (0, 1)$ of a given controlled invariant set $\mathcal{C}$. In particular, μ may be chosen arbitrarily close to 1 resulting in an arbitrarily close inner approximation of $\mathcal{C}$. The proposed method differs from the existing ones [3, 4] in that it permits, roughly speaking, the controlled system to first move away from the origin before moving towards it (where closeness to the origin is measured with the Minkowski function of $\mathcal{C}$). The existing methods, in contrast, use the assumed λ-contractiveness to design a controller that forces the controlled system closer to the origin by the factor λ in every time step, and therefore are more restrictive. The proposed controller can be understood as a special MPC scheme with prediction horizon fixed to n, where n denotes the dimension of the system.

We state notation and preliminaries in Sect. 2. The main result of the paper, i.e., the design of stabilizing control laws for linear constrained systems on controlled invariant sets, is presented in Sect. 3. Finally, we analyze a numerical example and state conclusions in Sects. 4 and 5, respectively.

2 Notation and preliminaries

We denote matrices by capital letters, vectors and scalars by lowercase letters and sets by calligraphic letters. Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ with $n, m \in \mathbb{N}$. By B^T, $\mathrm{rk}(B)$, $\sigma_{\min}(B)$ and $\sigma_{\max}(B)$ denote the transpose, the rank, the smallest and the largest singular value of B, respectively. Define $S_\nu(A, B) := (A^0B,\, A^1B, \ldots,\, A^{\nu-1}B)$ for $\nu \in \mathbb{N}_+$, where $\mathbb{N}_+ := \{i \in \mathbb{N} \,|\, i > 0\}$. Moreover, let $\mathbb{N}_{[j,k]} := \{i \in \mathbb{N} \,|\, j \le i \le k\}$ and denote nonnegative and positive reals with $\mathbb{R}_0$ and $\mathbb{R}_+$, respectively. By $\mathcal{B}^n(r) = \{x \in \mathbb{R}^n \,|\, \|x\|_2 \le r\}$, denote a ball in $\mathbb{R}^n$ with radius $r \in \mathbb{R}_+$. For an arbitrary set $\mathcal{C} \subset \mathbb{R}^n$, let $\partial\mathcal{C}$, $\mathrm{int}(\mathcal{C})$, and $\mathrm{cl}(\mathcal{C})$ refer to the boundary, the interior, and the closure of the set $\mathcal{C}$, respectively. Furthermore, for any $\lambda \in \mathbb{R}$, let $\lambda\,\mathcal{C} := \{\lambda\, x \,|\, x \in \mathcal{C}\}$. A convex and compact set $\mathcal{C} \subset \mathbb{R}^n$ with $0 \in \mathrm{int}(\mathcal{C})$ is called a C-set. Given a C-set $\mathcal{C}$, the function $\Psi_{\mathcal{C}} : \mathbb{R}^n \to \mathbb{R}_0$ with $\Psi_{\mathcal{C}}(x) := \inf\{\lambda \in \mathbb{R}_0 \,|\, x \in \lambda\,\mathcal{C}\}$

is called Minkowski function of $\mathcal{C}$ [5, p. 80]. We say a function $\alpha : \mathbb{R}_0 \to \mathbb{R}_0$ is of class $\mathcal{K}$ if α is continuous, strictly increasing and $\alpha(0) = 0$. Finally, let $\varrho : \mathbb{R}^n \to \mathbb{R}^m$ be a control law (resp. feedback law). The state of the *controlled system*

$$x(k+1) = A\,x(k) + B\,\varrho(x(k)), \qquad x(0) = x_0, \tag{3}$$

at time $k \in \mathbb{N}$ for initial condition x_0 is denoted by $\varphi(k, x_0, \varrho)$.

2.1 Basic definitions and assumptions

We intend to design *admissible* control laws that comply with the input and state constraints.

Definition 1: *Let $\mathcal{X}$ and $\mathcal{U}$ be the state and input constraints of system* (1)*, let $\mathcal{C} \subseteq \mathcal{X}$ with $0 \in \mathrm{int}(\mathcal{C})$ and let $\varrho : \mathcal{C} \to \mathbb{R}^m$. We call ϱ an admissible control law for* (1)*, if*
(i) $\varrho(0) = 0$,
(ii) $A\,x + B\,\varrho(x) \in \mathcal{C}$ *for every* $x \in \mathcal{C}$,
(iii) $\varrho(x) \in \mathcal{U}$ *for every* $x \in \mathcal{C}$,

Note that (i) implies that the origin is an equilibrium of the controlled system (3). Conditions (ii) and (iii) guarantee that the controlled system (3) satisfies the state and input constraints. In fact, (ii) and (iii) imply $\varphi(k, x_0, \varrho) \in \mathcal{C} \subseteq \mathcal{X}$ and $\varrho(\varphi(k, x_0, \varrho)) \in \mathcal{U}$ for every $x_0 \in \mathcal{C}$ and every $k \in \mathbb{N}$. We stress that admissibility of a control law $\varrho : \mathcal{C} \to \mathbb{R}^m$ is always linked to the domain $\mathcal{C}$ of ϱ. The notion of admissible control laws leads to the definition of stabilizing control laws.

Definition 2: *Let $\mathcal{D} \subseteq \mathcal{C} \subseteq \mathcal{X}$ with $0 \in \mathrm{int}(\mathcal{D})$ and let $\varrho : \mathcal{C} \to \mathbb{R}^m$ be an admissible control law for* (1)*. Then, ϱ is called stabilizing control law for* (1) *on $\mathcal{D}$, if*
(i) *the equilibrium* 0 *of* (3) *is asymptotically stable.*
(ii) *the set $\mathcal{D}$ is a subset of the domain of attraction, i.e., $\lim_{k\to\infty} \varphi(k, x_0, \varrho) = 0$ for every $x_0 \in \mathcal{D}$.*

It remains to recall the definitions of λ-contractive and controlled invariant C-sets.

Definition 3: *Let $\mathcal{C} \subseteq \mathcal{X}$ with $0 \in \mathrm{int}(\mathcal{C})$ and let $\lambda \in [0, 1]$. $\mathcal{C}$ is called λ-contractive, if, for every $x \in \mathcal{C}$, there exists a $u \in \mathcal{U}$ such that*

$$A\,x + B\,u \in \lambda\mathcal{C}. \tag{4}$$

For the special case $\lambda = 1$, a λ-contractive set $\mathcal{C}$ is also called controlled invariant.

The stabilizing control laws described in Sects. 2.2 and 3 build on the solution of optimization problems. We will frequently make use of the expressions introduced in the following definition.

Definition 4: *Consider the optimization problem*

$$\min_z f(z, \xi) \quad s.t. \quad g(z, \xi) \le 0, \quad h(z, \xi) = 0, \tag{5}$$

with variables $z \in \mathbb{R}^a$ and parameters $\xi \in \mathbb{R}^b$, where $f : \mathbb{R}^a \times \mathbb{R}^b \to \mathbb{R}$, $g : \mathbb{R}^a \times \mathbb{R}^b \to \mathbb{R}^c$, $h : \mathbb{R}^a \times \mathbb{R}^b \to \mathbb{R}^d$ and $a, b, c, d \in \mathbb{N}_+$. We call $\hat{z} \in \mathbb{R}^a$ feasible for (5) *at ξ, if $g(\hat{z}, \xi) \le 0$ and $h(\hat{z}, \xi) = 0$. The problem* (5) *is said to be feasible at ξ, if there exists at least one feasible $\hat{z} \in \mathbb{R}^a$ for* (5) *at ξ. We call $z^* \in \mathbb{R}^a$ optimal for* (5) *at ξ, if $z^* \in \mathbb{R}^a$ is feasible for* (5) *at ξ and if there does not exists any feasible $\hat{z} \neq z^*$ for* (5) *at ξ such that $f(\hat{z}, \xi) < f(z^*, \xi)$. We call $f(z^*, \xi)$ the solution to* (5) *at ξ.*

Finally, we make the following assumptions throughout the paper.

Assumption 1: *$\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{U} \subset \mathbb{R}^m$ are C-sets,* $\mathrm{rk}(B) = m$*, and* $\mathrm{rk}(S_n(A,B)) = n$*, where $m, n \in \mathbb{N}_+$.*

Note that $\mathrm{rk}(S_n(A,B)) = n$ implies that the pair (A, B) is controllable [9]. Further note that $\mathrm{rk}(B) = m$ implies $n \geq m$ and $B\,u = 0$ only if $u = 0$ [10, pp. 13-14].

2.2 Established results for λ-contractive sets

We briefly recall known results for λ-contractive C-sets. We begin by collecting some important properties of Minkowski functions (cf. [5, Prop. 3.12] and [16, Lem. 2.28]).

Lemma 1: *Let $\mathcal{C} \subset \mathbb{R}^n$ be a C-set and let $\Psi_\mathcal{C}$ be the associated Minkowski function. Let $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}_0$ and let $\underline{r}, \overline{r} \in \mathbb{R}_+$ be such that $\mathcal{B}^n(\underline{r}) \subseteq \mathcal{C} \subseteq \mathcal{B}^n(\overline{r})$. Then*
(i) *$\Psi_\mathcal{C}$ is convex on $\mathbb{R}^n$,*
(ii) $0 \leq \Psi_\mathcal{C}(x) < \infty$,
(iii) *$\Psi_\mathcal{C}(x) \leq \lambda$ if and only if $x \in \lambda\,\mathcal{C}$,*
(iv) $\overline{r}^{-1}\|x\|_2 \leq \Psi_\mathcal{C}(x) \leq \underline{r}^{-1}\|x\|_2$.

The following lemma, which immediately follows from Def. 3 (cf. [4, Thm. 3.3] and the subsequent discussion in [4]), guarantees the existence of an admissible control law associated with λ-contractive sets.

Lemma 2: *Let $\lambda \in [0,1]$ and let $\mathcal{C} \subseteq \mathcal{X}$ be a λ-contractive C-set. Then, there exists an admissible control law $\varrho : \mathcal{C} \to \mathbb{R}^m$ for (1), such that*

$$\Psi_\mathcal{C}(A\,x + B\,\varrho(x)) \leq \lambda\,\Psi_\mathcal{C}(x) \tag{6}$$

for every $x \in \mathcal{C}$.

For $\lambda < 1$, relation (6) guarantees that every state $x \in \mathcal{C}$ can be moved closer to the origin within one time step (where closeness to the origin is measured by the Minkowski function value). In other words, for $\lambda < 1$, Lem. 2 guarantees the existence of a stabilizing control law for (1) on $\mathcal{C}$. To specify this control law, consider the optimization problem

$$\begin{aligned} \min_{u(0),x(1)} \quad & \Psi_\mathcal{C}(x(1)) + \Psi_\mathcal{C}(\xi) \\ \text{s.t.} \quad & x(1) = A\,\xi + B\,u(0), \\ & \Psi_\mathcal{C}(x(1)) \leq 1, \\ & \Psi_\mathcal{U}(u(0)) \leq 1, \end{aligned} \tag{7}$$

with variables[1] $z = (u(0)^T\ x(1)^T)^T \in \mathbb{R}^{m+n}$ and parameters $\xi \in \mathcal{X}$. Note that ξ will later be identified with the current state of the system in order to design a feedback law. We summarize some important properties of (7) in the following remark.

Remark 1: *For every $\xi \in \mathbb{R}^n$, (7) is a finite-dimensional convex optimization problem [6, pp. 136–137], due to convexity of the Minkowski functions $\Psi_\mathcal{C}$ and $\Psi_\mathcal{U}$ (see Lem. 1 (i)), since the equality constraints are linear, and since level sets of convex functions describe convex sets [1, Lem. 3.1.2]. Moreover, if $\mathcal{C}$ is a λ-contractive C-set (with $\lambda \in [0,1]$), (7) is feasible at every $\xi \in \mathcal{C}$, since the existence of a $\hat{u}(0) \in \mathcal{U}$ (i.e., $\Psi_\mathcal{U}(u(0)) \leq 1$) such that*

[1] Note that the variable $x(1)$ can easily be eliminated from (7), since it is uniquely determined by $u(0)$ and ξ. We include it, however, for better readability.

$\hat{x}(1) = A\,\xi + B\,\hat{u}(0) \in \mathcal{C}$ *(i.e.,* $\Psi_{\mathcal{C}}(x(1)) \leq 1$*) is guaranteed for every* $\xi \in \mathcal{C}$ *by definition of* λ*-contractive sets (see Def. 3). Finally, for polytopic sets* $\mathcal{U}$ *and* $\mathcal{C}$*, (7) can be written as a multiparametric linear program which can be solved efficiently (see, e.g., [6, p. 146] and [11]).*

If $\mathcal{C}$ is a λ-contractive C-set with $\lambda < 1$, a stabilizing control law $\varrho : \mathcal{C} \to \mathbb{R}^m$ for (1) on $\mathcal{D} = \mathcal{C}$ is given by

$$\varrho(x) = u^*(0) = (\,I_m\;\;0\,)\,z^*, \tag{8}$$

where $z^* = (u^*(0)^T\;\,x^*(1)^T)^T$ is optimal for (7) at the current state $\xi = x$ (cf. [4, Thm. 3.3] and [5, p. 139]). To see this, first note that (8) is an admissible control law for (1) due to the constraints in (7), due to feasibility of (7) for every $\xi \in \mathcal{C}$ (see Rem. 1), and since $\varrho(0) = 0$ can be guaranteed based on $\mathrm{rk}(B) = m$ (see Assum. 1). Moreover, it is straightforward to show that the function $v : \mathcal{C} \to \mathbb{R}_0$, where $v(x)$ refers to the solution to (7) at $\xi = x$, is a Lyapunov function of the controlled system (3). In fact, there exist functions[2] α, β, γ of class $\mathcal{K}$ such that

$$\alpha(\|\varphi(k, x_0, \varrho)\|_2) \leq v(\varphi(k, x_0, \varrho)) \leq \beta(\|\varphi(k, x_0, \varrho)\|_2), \tag{9}$$

$$v(\varphi(k+1, x_0, \varrho)) \leq v(\varphi(k, x_0, \varrho)) - \gamma(\|\varphi(k, x_0, \varrho)\|_2) \tag{10}$$

for every $x_0 \in \mathcal{D} = \mathcal{C}$ and all $k \in \mathbb{N}$.

3 Stabilizing controlled invariant sets

In this section, we present a new method for the systematic design of stabilizing control laws for (1) on controlled invariant sets. We first motivate the new approach by showing that the controller defined by (7) and (8) in Sect. 2.2 may fail for controlled invariant sets that are not λ-contractive for any $\lambda \in [0, 1)$.

3.1 Motivating example

Consider system (1) with $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$ and constraints $\mathcal{X} = \{x \in \mathbb{R}^2 \,|\, |x_1| \leq 5,\, |x_2| \leq 5\}$ and $\mathcal{U} = \{u \in \mathbb{R} \,|\, |u| \leq 1\}$. We show that $\mathcal{C} = \mathcal{X}$ is controlled invariant but not λ-contractive for any $\lambda \in [0, 1)$ in Sect. 4. We first ignore this result and attempt to apply the controller from Sect. 2.2. Specifically, we consider the initial state $x_0 = (\,-3\;\;3\,)^T \in \mathcal{C}$ and try to steer x_0 to the origin. We claim without giving details that solving (7) for $\xi = x_0$ yields $u^*(0) = 0$ and $x^*(1) = (\,3\;\;-3\,)^T$. (See the dash-dotted one-step reachable set in Fig. 1 for a geometrical motivation of this solution.) Thus, the application of control law (8) results in

$$\varphi(1, x_0, \varrho) = A\,x_0 + B\,\varrho(x_0) = A\,x_0 + B\,u^*(0) = x^*(1).$$

Solving (7) again for $\xi = \varphi(1, x_0, \varrho)$ yields $u^*(0) = 0$ and $x^*(1) = (\,-3\;\;3\,)^T$. Obviously, $\varphi(2, x_0, \varrho) = x_0$. In fact, the trajectory of the controlled system (3) enters a limit cycle between the states x_0 and $(\,3\;\;-3\,)^T$ and consequently never reaches the origin (see Fig. 1).

However, it is possible to steer x_0 towards the origin. The choices $u(0) = 1$ and $u(1) = -1$ result in $x(1) = (\,4\;\;-1\,)^T$ and $x(2) = (\,-2\;\;2\,)^T$. Clearly, the trajectory has moved closer to the origin (see Fig. 1). Continuing with $u(2) = u(4) = 1$ and $u(3) = u(5) = -1$ finally yields $x(6) = (\,0.0\;\;0.0\,)^T$.

[2] Consider, e.g., $\alpha(z) = \frac{1}{\overline{r}_x} z$, $\beta(z) = \frac{1+\lambda}{\underline{r}_x} z$, and $\gamma(z) = \frac{1-\lambda^2}{\overline{r}_x} z$, where $\underline{r}_x$ and $\overline{r}_x$ are as in Prop. 1. Note that γ is of class $\mathcal{K}$ since $\lambda < 1$.

The example shows the method summarized in Sect. 2 does in general not result in a stabilizing control law on controlled invariant sets. The initial state x_0 used in the example can, however, be steered towards the origin. It remains to answer the question whether there exists a simple stabilizing control law similar to (8) that applies to controlled invariant sets.

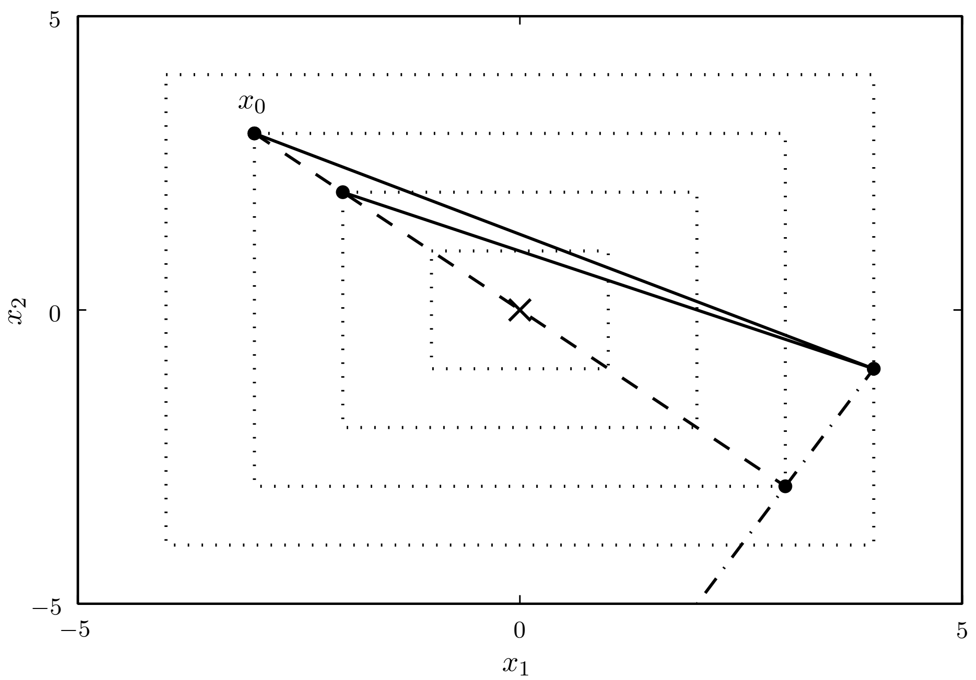

Figure 1: Trajectories of the example for the initial condition $x_0 = (\,-3\;\;3\,)^T$. The ranges $-5 \leq x_1, x_2 \leq 5$ correspond to the set $\mathcal{X} = \mathcal{C}$. The dashed line connects the states of the limit cycle that results for the controller from (8) (see Sect. 2.2). The solid line connects the states that result for $u(0) = 1$ and $u(1) = -1$. The dash-dotted line represents the set of states that can be reached from x_0 in one step for $u(0) \in \mathcal{U}$. The dotted rectangles refer to level sets of the Minkowski function $\Psi_{\mathcal{C}}(x)$.

3.2 Stabilizing control scheme with n-step prediction

Consider the optimization problem

$$\begin{aligned} \min_{\substack{u(0),\dots,u(n-1),\\ x(0),\dots,x(n)}} \quad & f(z,\xi) \\ \text{s.t.} \quad & x(0) = \xi, \\ & \left.\begin{aligned} & x(k+1) = A\,x(k) + B\,u(k), \\ & \Psi_{\mathcal{C}}(x(k+1)) \leq 1, \\ & \Psi_{\mathcal{U}}(u(k)) \leq 1, \end{aligned}\right\} k \in \mathbb{N}_{[0,n-1]} \end{aligned} \tag{11}$$

with variables[3]

$$z = (u(0)^T \dots\, u(n-1)^T\;\, x(0)^T \dots\, x(n)^T)^T \in \mathbb{R}^{(m+n+1)\,n},$$

[3] Variables $x(0), \dots, x(n)$ can be easily eliminated from (11) with the same argument as in footnote 1.

parameters $\xi \in \mathcal{X}$, and the objective function

$$f(z,\xi) = w\,\Psi_{\mathcal{C}}(x(n)) + \textstyle\sum_{k=0}^{n-1} \Psi_{\mathcal{C}}(x(k)), \tag{12}$$

where $w \in \mathbb{R}_+$ denotes a weight. Note that (11) differs from (7) in that it includes n prediction steps instead of one, where n refers to the system dimension. Moreover, in contrast to (7), the Minkowski function value of the last step is weighted by w in (12). However, the optimization problems (7) and (11) are similar in terms of the traits summarized in Rem. 1. In fact, (11) is also convex and feasible for every $\xi \in \mathcal{C}$. In addition, (11) can also be written as a multiparametric linear program for polytopic sets $\mathcal{U}$ and $\mathcal{C}$.

The following proposition states the main result of the paper. Basically, it provides a method for the design of stabilizing control laws on controlled invariant sets. Note that there is no assumption on λ-contractivity.

Proposition 1: *Let $\mathcal{C} \subseteq \mathcal{X}$ be a controlled invariant C-set and let $\mu \in (0,1)$ be arbitrary. Then, there exists a stabilizing control law for (1) on $\mathcal{D} = \mu\,\mathcal{C}$, which can be constructed as follows. Let $\underline{r}_u, \underline{r}_x, \overline{r}_x \in \mathbb{R}_+$ be such that*

$$\mathcal{B}^m(\underline{r}_u) \subseteq \mathcal{U}, \qquad \mathcal{B}^n(\underline{r}_x) \subseteq \mathcal{C} \subseteq \mathcal{B}^n(\overline{r}_x) \tag{13}$$

and

$$\underline{r}_x \geq \sqrt{n}\,\sigma_{\max}(S_n(A,B))\,\underline{r}_u. \tag{14}$$

Define

$$\tilde{\lambda} := 1 - \sigma_{\min}(S_n(A,B))\,\frac{\underline{r}_u}{\overline{r}_x}\,\min\left\{\frac{1}{\mu} - 1, 1\right\} \tag{15}$$

and choose $w \in \mathbb{R}_+$ such that

$$\left(\frac{2-\tilde{\lambda}}{1-\tilde{\lambda}}\,\frac{1}{\max(\mu, 1-\mu)} - 1\right)(n-1) < w < \infty. \tag{16}$$

Then, $\varrho : \mathcal{C} \to \mathbb{R}^m$ with

$$\varrho(x) = u^*(0) = (\,I_m \;\; 0\,)\,z^*, \tag{17}$$

where $z^* = (u^*(0)^T, \ldots, u^*(n-1)^T, x^*(0)^T, \ldots, x^*(n)^T)^T$ is optimal for (11) at $\xi = x$, is a stabilizing control law for (1) on $\mathcal{D} = \mu\,\mathcal{C}$.

Proposition 1 can be used to design stabilizing control laws for (1) on $\mathcal{D} = \mu\,\mathcal{C}$ for every $\mu \in (0,1)$. It is necessary to exclude $\partial\mathcal{C}$ (i.e., $\mu = 1$), since there may be unstabilizable states on the boundary of controlled invariant sets. In fact, revisiting the motivating example in Sect. 3.1, it is easy to show that the initial states $x_0 = (\,-5\;\;5\,)^T \in \partial\mathcal{C}$ and $x_0 = (\,5\;\;-5\,)^T \in \partial\mathcal{C}$ cannot be asymptotically stabilized, since they cannot be steered closer to the origin using any admissible control law. To see this, note that for both initial states x_0, the successor $Ax_0 + B\,u$ violates the state constraints $\mathcal{X} = \mathcal{C}$ for every $u \in \mathcal{U}\backslash\{0\}$. However, for the only admissible choice $u = 0 \in \mathcal{U}$, we obtain $A\,x_0 + B\,u = Ax_0 = -x_0$, i.e., we enter a limit cycle between the states $(\,-5\;\;5\,)^T$ and $(\,5\;\;-5\,)^T$.

3.3 Formal proof of Proposition 1

The proof of Prop. 1 requires some preparation. Analogously to the argumentation in Sect. 2.2, we first note that the control law $\varrho : \mathcal{C} \to \mathbb{R}^m$ as defined in Prop. 1 is an admissible control law for (1). To prove that (17) is a stabilizing control law for (1) on $\mathcal{D} = \mu\,\mathcal{C}$, we will show that the solution to (11) is a Lyapunov function of the controlled system (3). We begin by collecting some statements about $\underline{r}_u, \underline{r}_x, \overline{r}_x, \tilde{\lambda}$, and w introduced in Prop. 1. Obviously, since $\mathcal{U}$ and $\mathcal{C}$ are C-sets by assumption, there always exist appropriate $\underline{r}_u, \underline{r}_x, \overline{r}_x \in \mathbb{R}_+$ such that relations (13) and (14) hold. Concerning $\tilde{\lambda}$ and w, we find the bounds stated in Lem. 3. We omit the simple proof due to space limitations.

Lemma 3: *Let $\tilde{\lambda}$ and w be as in* (15) *and* (16)*, respectively. Then,* $\tilde{\lambda} \in [0, 1)$ *and* $w > n-1$.

Note that, for the special case $n = 1$, the choice $w = 1$ is such that (16) holds. In this case, the optimization problems (7) and (11) as well as the associated control laws (8) and (17) become identical. In contrast, for any $n > 1$, we require $w > 1$ according to Lem. 3.

Lemma 4 provides a first characterization of the solution to (11). It basically states that optimal variables for (11) at $\xi \in \mathcal{C}$ are always such that the states $x^*(k)$, $k \in \mathbb{N}_{[0,n-1]}$, are not closer to the origin than $x^*(n)$, where closeness to the origin is again measured by the Minkowski function value.

Lemma 4: *Let $\mathcal{C} \subseteq \mathcal{X}$ be a controlled invariant C-set. Let $\xi \in \mathcal{C}$ be arbitrary and let $z^* = (u^*(0)^T, \ldots, u^*(n-1)^T, x^*(0)^T, \ldots, x^*(n)^T)^T$ be optimal for* (11) *at ξ. Then,*

$$\Psi_{\mathcal{C}}(x^*(n)) = \min_{k \in \mathbb{N}_{[0,n]}} \Psi_{\mathcal{C}}(x^*(k)). \tag{18}$$

Proof. We show that, if (18) does not hold, there exists a feasible

$$\hat{z} = [\hat{u}(0)^T, \ldots, \hat{u}(n-1)^T, \hat{x}(0)^T, \ldots, \hat{x}(n)^T]^T$$

for (11) at ξ such that

$$f(\hat{z}, \xi) < f(z^*, \xi), \tag{19}$$

where f refers to the objective function (12) of (11). According to Def. 4, (19) implies that z^* is not optimal for (11) at ξ, which is a contradiction.

If (18) does not hold, there exists an $s \in \mathbb{N}_{[0,n-1]}$ such that

$$\Psi_{\mathcal{C}}(x^*(s)) < \Psi_{\mathcal{C}}(x^*(n)) \qquad \text{and} \tag{20}$$

$$\Psi_{\mathcal{C}}(x^*(s)) \leq \Psi_{\mathcal{C}}(x^*(k)) \qquad \text{for } k \in \mathbb{N}_{[0,n-1]}. \tag{21}$$

Thus, we find $\Psi_{\mathcal{C}}(x^*(s)) \leq \Psi_{\mathcal{C}}(x^*(0)) = \Psi_{\mathcal{C}}(\xi)$, which in turn implies $x^*(s) \in \mathcal{C}$ according to Lem. 1. Since $\mathcal{C}$ is controlled invariant, there exists an admissible control law $\hat{\varrho}$ for (1) that fulfills (6) with $\lambda = 1$. Applying this control law from time s yields

$$\hat{x}(k) = \begin{cases} \varphi(k-s, x^*(s), \hat{\varrho}) & \text{if} \quad k \geq s \\ x^*(k) & \text{otherwise} \end{cases} \tag{22}$$

for every $k \in \mathbb{N}_{[0,n]}$ and

$$\hat{u}(k) = \begin{cases} \hat{\varrho}(\varphi(k-s, x^*(s), \hat{\varrho})) & \text{if} \quad k \geq s \\ u^*(k) & \text{otherwise} \end{cases} \tag{23}$$

for every $k \in \mathbb{N}_{[0,n-1]}$. Clearly, $\hat{z}$ with $\hat{x}(k)$ and $\hat{u}(k)$ as in (22) and (23), respectively, is feasible for (11) at ξ. It immediately follows from (6) with $\lambda = 1$ that

$$\Psi_{\mathcal{C}}(\hat{x}(k)) = \Psi_{\mathcal{C}}(\varphi(k-s, x^*(s), \hat{\varrho})) \leq \Psi_{\mathcal{C}}(x^*(s)) \tag{24}$$

for every $k \in \mathbb{N}_{[s,n]}$. Thus, we find

$$\begin{aligned} & f(\hat{z}, \xi) = w\, \Psi_{\mathcal{C}}(\hat{x}(n)) + \textstyle\sum_{k=0}^{n-1} \Psi_{\mathcal{C}}(\hat{x}(k)) \\ & \leq w\, \Psi_{\mathcal{C}}(x^*(s)) + \textstyle\sum_{k=s}^{n-1} \Psi_{\mathcal{C}}(x^*(s)) + \sum_{k=0}^{s-1} \Psi_{\mathcal{C}}(x^*(k)) \\ & \leq w\, \Psi_{\mathcal{C}}(x^*(s)) + \textstyle\sum_{k=s}^{n-1} \Psi_{\mathcal{C}}(x^*(k)) + \sum_{k=0}^{s-1} \Psi_{\mathcal{C}}(x^*(k)) \\ & < w\, \Psi_{\mathcal{C}}(x^*(n)) + \textstyle\sum_{k=0}^{n-1} \Psi_{\mathcal{C}}(x^*(k)) = f(z^*, \xi) \end{aligned}$$

where the second, third, and fourth relation hold due to (22) in combination with (24), (21), and (20). In summary, we obtain the contradiction (19). ■

In the remainder of this section, we will prove that the function $v : \mathcal{C} \to \mathbb{R}_0$, where

$$v(x) := f(z^*, \xi) \tag{25}$$

for every $x \in \mathcal{C}$ and where z^* solves (11) for $\xi = x$, is a Lyapunov function for the controlled system (3). We will show that there exist functions α, β, γ of class $\mathcal{K}$ such that relations (9) and (10) hold for every $x_0 \in \mathcal{D} = \mu\,\mathcal{C}$ and all $k \in \mathbb{N}$. A necessary condition for (10) to hold is stated in Lem. 5.

Lemma 5: *Let $\mathcal{C} \subseteq \mathcal{X}$ be a controlled invariant C-set and let $\varrho : \mathcal{C} \to \mathbb{R}^m$ and $v : \mathcal{C} \to \mathbb{R}_0$ be defined as in Prop. 1 and Eq. (25), respectively. Then, for every $x_0 \in \mathcal{C}$, we have*

$$v(\varphi(k+1, x_0, \varrho)) \leq v(\varphi(k, x_0, \varrho)). \tag{26}$$

Proof. Initially note that $\varphi(k, x_0, \varrho) \in \mathcal{C}$ for every $x_0 \in \mathcal{C}$ and every $k \in \mathbb{N}$ since ϱ is an admissible control law on $\mathcal{C}$. Now, let $z^* = (u^*(0)^T \dots u^*(n-1)^T \, x^*(0)^T \dots x^*(n)^T)^T$ be optimal for (11) at $\xi = \varphi(k, x_0, \varrho)$. We prove (26) by showing that there exists a feasible $\hat{z} = (\hat{u}(0)^T \dots \hat{u}(n-1)^T \, \hat{x}(0)^T \dots \hat{x}(n)^T)^T$ for (11) at $\xi = \varphi(k+1, x_0, \varrho)$ such that

$$f(\hat{z}, \varphi(k+1, x_0, \varrho)) \leq f(z^*, \varphi(k, x_0, \varrho)). \tag{27}$$

Since $\mathcal{C}$ is a controlled invariant C-set, there exists an admissible control law $\hat{\varrho} : \mathcal{C} \to \mathbb{R}^m$ for (1) such that (6) holds with $\lambda = 1$ for every $x \in \mathcal{C}$. Now, let $\hat{u}(k) := u^*(k+1)$ for every $k \in \mathbb{N}_{[0,n-2]}$ and $\hat{u}(n-1) := \hat{\varrho}(x^*(n))$. Analogously, let $\hat{x}(k) := x^*(k+1)$ for every $k \in \mathbb{N}_{[0,n-1]}$ and $\hat{x}(n) := A\,x^*(n) + B\,\hat{\varrho}(x^*(n))$. Note that $\hat{z}$ is feasible for (11) at $\varphi(k+1, x_0, \varrho) = x^*(1)$. Evaluating (27) with f as in (12) yields

$$w\,\Psi_{\mathcal{C}}(A\,x^*(n) + \hat{\varrho}(x^*(n))) \leq (w-1)\,\Psi_{\mathcal{C}}(x^*(n)) + \Psi_{\mathcal{C}}(x^*(0)).$$

This relation holds, since $\Psi_{\mathcal{C}}(A\,x^*(n) + \hat{\varrho}(x^*(n))) \leq \Psi_{\mathcal{C}}(x^*(n))$ according to Eq. (6) and since $\Psi_{\mathcal{C}}(x^*(n)) \leq \Psi_{\mathcal{C}}(x^*(0))$ according to Lem. 4. ■

The following lemma provides the key to prove Prop. 1. It basically provides an upper bound for the solution to (11) at any point on a trajectory of the controlled system (3) with initial condition $x_0 \in \mathcal{D} = \mu\,\mathcal{C}$.

Lemma 6: *Let $\mathcal{C} \subseteq \mathcal{X}$ be a controlled invariant C-set and let $\varrho : \mathcal{C} \to \mathbb{R}^m$ and $v : \mathcal{C} \to \mathbb{R}_0$ be defined as in Prop. 1 and Eq. (25), respectively. Let $\mu \in (0,1)$, $k \in \mathbb{N}$, and $x_0 \in \mathcal{D} = \mu\,\mathcal{C}$ be arbitrary and set $\xi = \varphi(k, x_0, \varrho)$. Then,*

$$v(\xi) \leq (2\,n - 1 + \tilde{\lambda} w)\,\Psi_{\mathcal{C}}(\xi), \tag{28}$$

where $\tilde{\lambda}$ and w are defined as in Prop. 1.

Proof. First note that $\xi = \varphi(k, x_0, \varrho) \in \mathcal{C}$ for every $x_0 \in \mathcal{D} = \mu\,\mathcal{C}$ since ϱ as defined in (17) is admissible for (1). The proof consists of two parts that address the cases $\xi = \varphi(k, x_0, \varrho) \in \mu\,\mathcal{C}$ and $\varphi(k, x_0, \varrho) \in \mathcal{C} \setminus \mu\,\mathcal{C}$, respectively. Throughout the proof, let $\underline{r}_u$, $\underline{r}_x$ and $\overline{r}_x$ be as in Prop. 1.

Part I ($\xi \in \mu\,\mathcal{C}$). According to [**?**, Lem. 3.47], there exists a feasible $\hat{z} = (\hat{u}(0)^T \dots \hat{u}(n-1)^T \, \hat{x}(0)^T \dots \hat{x}(n)^T)^T$ for (11) at $\xi \in \mu\,\mathcal{C}$ such that $\Psi_{\mathcal{C}}(\hat{x}(k)) \leq 2\,\Psi_{\mathcal{C}}(\xi)$ for every $k \in \mathbb{N}_{1,n-1}$ and such that

$$\Psi_{\mathcal{C}}(\hat{x}(n)) \leq \tilde{\lambda}\,\Psi_{\mathcal{C}}(\xi), \tag{29}$$

where $\tilde{\lambda}$ is defined as in Prop. 1. Taking $\hat{x}(0) = \xi$ into account and overestimating the associated objective function value $f(\hat{z}, \xi)$ (with f as in (12)) yields

$$f(\hat{z}, \xi) \leq (\tilde{\lambda} w + 2\,(n-1) + 1)\, \Psi_{\mathcal{C}}(\xi),$$

which proves (28).

Part II ($\xi \in \mathcal{C} \setminus \mu\,\mathcal{C}$). First note that $\xi = \varphi(k, x_0, \varrho) \in \mathcal{C} \setminus \mu\,\mathcal{C}$ implies $k \geq 1$ since $x_0 = \varphi(0, x_0, \varrho) \in \mu\,\mathcal{C}$ by assumption. Furthermore, we have $\Psi_{\mathcal{C}}(x_0) \leq \mu < \Psi_{\mathcal{C}}(\xi)$ according to Lem. 1 (iii), since $x_0 \in \mu\,\mathcal{C}$ and since $\xi \notin \mu\,\mathcal{C}$. Finally, we obtain

$$v(\xi) \leq v(\varphi(k-1, x_0, \varrho)) \leq \dots \leq v(x_0) \tag{30}$$

according to Lem. 5. Since $x_0 \in \mu\,\mathcal{C}$, we have

$$v(x_0) \leq (2\,n - 1 + \tilde{\lambda} w)\, \Psi_{\mathcal{C}}(x_0) \tag{31}$$

according to the first part of this proof. Combining (30) and (31) and using $\Psi_{\mathcal{C}}(x_0) < \Psi_{\mathcal{C}}(\xi)$ proves (28). ■

Equation (29) states that any $x_0 \in \mathcal{D} = \mu\,\mathcal{C}$ can be steered closer to the origin in n steps, where closeness to the origin is measured by the Minkowski function value. It remains to answer the question whether the optimum of (11) enforces a movement towards the origin. Lemma 7 stated below implies that a contraction is always guaranteed. Note that this result is not trivial. The motivating example in Sect. 3.1 shows that it may be necessary to move away from the origin first. Such a detour implies an increase of some terms in the objective function value (12). In fact, it is necessary to choose the weight w (see Eq. (16)) carefully in the objective function (12) to compensate for these effects.

Lemma 7: *Let $\mathcal{C} \subseteq \mathcal{X}$ be a controlled invariant C-set. Let $\mu \in (0,1)$, $k \in \mathbb{N}$ and $x_0 \in \mathcal{D} = \mu\,\mathcal{C}$ be arbitrary, set $\xi = \varphi(k, x_0, \varrho)$ and let $z^* = (u^*(0)^T \dots u^*(n-1)^T\, x^*(0)^T \dots x^*(n)^T)^T$ be optimal for* (11) *at ξ. Then,*

$$\Psi_{\mathcal{C}}(x^*(n)) \leq \lambda^*\, \Psi_{\mathcal{C}}(\xi), \tag{32}$$

where

$$\lambda^* := \tilde{\lambda} + (1 - \tilde{\lambda})\, \max\{\mu, 1-\mu\} < 1 \tag{33}$$

and where $\tilde{\lambda}$ is defined as in (15).

Proof. First note that we have $\lambda^* \in [0,1)$ since $\mu \in (0,1)$ and since $\tilde{\lambda} \in [0,1)$ according to Lem. 3. To prove (32), we assume

$$\Psi_{\mathcal{C}}(x^*(n)) > \lambda^*\, \Psi_{\mathcal{C}}(\xi), \tag{34}$$

and show that a contradiction results. We again have $\xi = \varphi(k, x_0, \varrho) \in \mathcal{C}$ with the same reason as in the proof of Lem. 6. Thus, if (34) holds, we find

$$\Psi_{\mathcal{C}}(x^*(k)) \geq \Psi_{\mathcal{C}}(x^*(n)) > \lambda^*\, \Psi_{\mathcal{C}}(\xi), \tag{35}$$

for every $k \in \mathbb{N}_{[0,n]}$ according to Lem. 4. Overestimating $v(\xi) = f(z^*, \xi)$ using (35) and $\Psi_{\mathcal{C}}(x^*(0)) = \Psi_{\mathcal{C}}(\xi)$ yields

$$v(\xi) > ((w + n - 1)\, \lambda^* + 1)\, \Psi_{\mathcal{C}}(\xi). \tag{36}$$

Note that the strict inequality in (36) follows for $w > 0$, which holds according to Lem. 3 since $n \geq 1$. On the other hand, we obtain

$$v(\xi) \leq (2\,n - 1 + \tilde{\lambda} w)\, \Psi_{\mathcal{C}}(\xi) \tag{37}$$

from Lem. 6. Clearly, the bounds (36) and (37) require $2\,n - 1 + \tilde{\lambda}\, w > (w + n - 1)\, \lambda^* + 1$ or equivalently

$$\frac{2 - \lambda^*}{\lambda^* - \tilde{\lambda}}\,(n-1) > w. \tag{38}$$

However, (38) contradicts (16). To see this, note that the l.h.s. in (16) and (38) are equivalent. In fact, we find

$$\frac{2 - \lambda^*}{\lambda^* - \tilde{\lambda}} = \frac{2 - \tilde{\lambda}}{1 - \tilde{\lambda}}\, \frac{1}{\max\{\mu, 1 - \mu\}} - 1,$$

by definition of λ^* in (33). ■

Lemmas 6 and 7 finally allow to prove Prop. 1.

Proof of Prop. 1. First note that $\mu\,\mathcal{C}$ is a C-set since $\mathcal{C}$ is a C-set by assumption and since $\mu \in (0,1)$. Thus, with regard to Def. 1, the control law $\varrho : \mathcal{C} \to \mathbb{R}^m$ as defined in (17) is not only admissible for (1) on $\mathcal{C}$ but also on $\mu\,\mathcal{C}$. Consequently, it remains to show that conditions (i) and (ii) in Def. 2 hold. Assume there exist functions α, β, γ of class $\mathcal{K}$ such that (9) and (10) hold. Then, it is easy to prove conditions (i) and (ii) based on standard arguments (see., e.g., [18, pp. 165-167 and p. 267]). In fact, the proof here is simpler than in [18], since $\varphi(k, x_0, \varrho)$ is bounded according to $\varphi(k, x_0, \varrho) \in \mathcal{C} \subseteq \mathcal{X}$ for every $k \in \mathbb{N}$ and every $x_0 \in \mu\,\mathcal{C}$.

It remains to construct appropriate functions $\alpha, \beta, \gamma : \mathbb{R}_0 \to \mathbb{R}_0$. Let $\alpha(z) := \frac{1}{\overline{r}_x}\, z$,

$$\beta(z) := \frac{2\,n - 1 + \tilde{\lambda}\, w}{\underline{r}_x}\, z \quad \text{and} \quad \gamma(z) := \frac{1 - \lambda^*}{\overline{r}_x}\, z, \tag{39}$$

where $\underline{r}_x$ and $\overline{r}_x$ satisfy the assumption in Prop. 1 and where $\tilde{\lambda}$, w and λ^* are defined as in (15), (16) and (33), respectively. Note that α, β and γ are of class $\mathcal{K}$, since $\underline{r}_x, \overline{r}_x, w \in \mathbb{R}_+$, due to $\tilde{\lambda}, \lambda^* \in [0,1)$ and since $n \in \mathbb{N}_+$. In the following, let $x_0 \in \mu\,\mathcal{C}$ and $k \in \mathbb{N}$ be arbitrary but fixed and set $\xi = \varphi(k, x_0, \varrho)$. To show (9), note that

$$\Psi_{\mathcal{C}}(\xi) \leq v(\xi) \leq (2\,n - 1 + \tilde{\lambda} w)\, \Psi_{\mathcal{C}}(\xi) \tag{40}$$

due to nonnegativeness of $\Psi_{\mathcal{C}}$ (see Lem. 1) and according to Lem. 6, respectively. Based on (40), we obtain

$$\alpha(\|\xi\|_2) = \frac{\|\xi\|_2}{\overline{r}_x} \leq \Psi_{\mathcal{C}}(\xi) \leq v(\xi), \tag{41}$$

where the first and the second relation hold by definition of α and according to Lem. 1, respectively. Obviously, (41) proves the first relation in (9). Analogously, it can be shown that the second relation in (9) holds for β as in (39).

To show (10), first note that, according to Lem. 2, there exists an admissible control law $\hat{\varrho}$ for (1) on $\mathcal{C}$ such that (6) holds with $\lambda = 1$. Now, let $z^* = (u^*(0)^T \ldots u^*(n-1)^T\, x^*(0)^T \ldots x^*(n)^T)^T$ be optimal for (11) at ξ. Set $\hat{u}(k) = u^*(k+1)$ for every $k \in \mathbb{N}_{[0,n-2]}$ and $\hat{u}(n-1) = \hat{\varrho}(x^*(n))$. Moreover, let $\hat{x}(k) = x^*(k+1)$ for every $k \in \mathbb{N}_{[0,n-1]}$ and $\hat{x}(n) := A\, x^*(n) + B\, \hat{\varrho}(x^*(n))$. Note that $\Psi_{\mathcal{C}}(\hat{x}(n)) \leq \Psi_{\mathcal{C}}(x^*(n))$ according to (6). Further note that

$\hat{z} = (\hat{u}(0)^T \dots \hat{u}(n-1)^T\, \hat{x}(0)^T \dots \hat{x}(n)^T)^T$ is feasible for (11) at $\varphi(1,\xi,\varrho) = \varphi(k+1, x_0, \varrho)$. Thus, we obtain $v(\varphi(1,\xi,\varrho)) \leq f(\hat{z}, \varphi(1,\xi,\varrho))$, where

$$\begin{aligned} f(\hat{z}, \varphi(1,\xi,\varrho)) &\leq w\,\Psi_{\mathcal{C}}(x^*(n)) + \textstyle\sum_{k=1}^{n} \Psi_{\mathcal{C}}(x^*(k)) \\ &= f(z^*, \xi) + \Psi_{\mathcal{C}}(x^*(n)) - \Psi_{\mathcal{C}}(\xi) \\ &\leq f(z^*, \xi) + (\lambda^* - 1)\,\Psi_{\mathcal{C}}(\xi). \end{aligned} \tag{42}$$

The first relation in (42) holds by construction of $\hat{x}(k)$, $k \in \mathbb{N}_{[0,n]}$. The second relation holds by definition of f in (12). Finally, the third relation is satisfied according to Lem. 7. Now, by definition, we have $v(\xi) = f(z^*, \xi)$. Thus, we infer

$$v(\varphi(1,\xi,\varrho)) - v(\xi) \leq (\lambda^* - 1)\,\Psi_{\mathcal{C}}(\xi). \tag{43}$$

It is easy to prove (10) with (43) and γ as in (39). ■

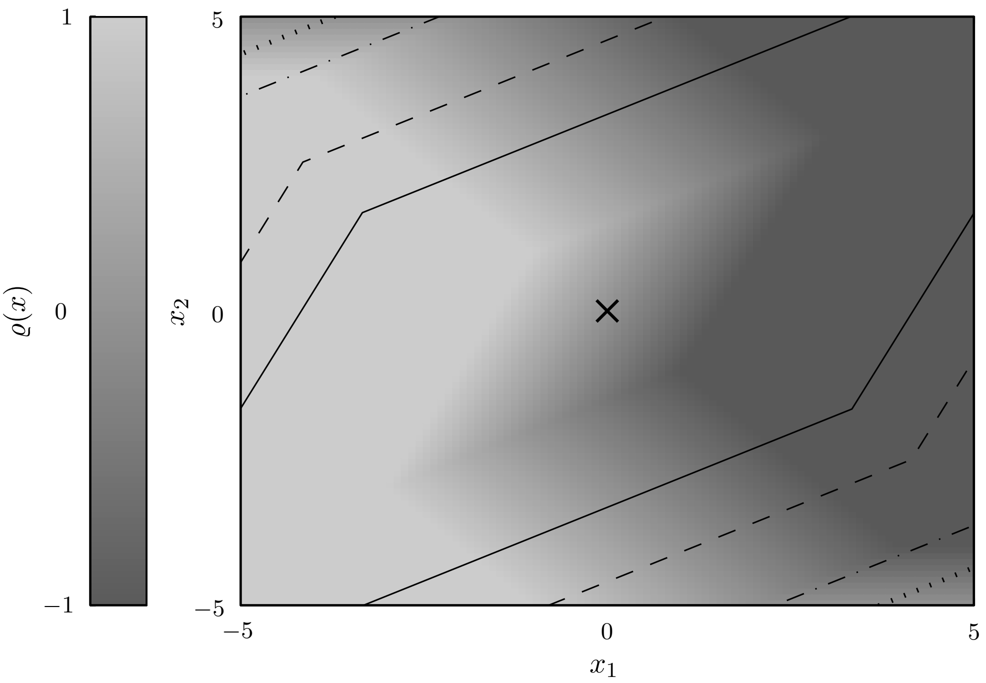

Figure 2: The proposed control law and λ-contractive sets for the sample system introduced in Sect. 3.1. The ranges $-1 \leq \varrho(x) \leq 1$ and $-5 \leq x_1, x_2 \leq 5$ correspond to the sets $\mathcal{U}$ and $\mathcal{X} = \mathcal{C}$, respectively. The solid, dashed, dash-dotted and dotted polytopes mark tight outer approximations of the largest λ-contractive subsets of $\mathcal{C}$ for $\lambda \in \{0.8, 0.85, 0.9, 0.95\}$, respectively. The colormap visualizes the contractive control law introduced in Prop. 1 for the choice $w = 708$.

4 Numerical example

Consider the system described in Sect. 3.1 again. We anticipated that the C-set $\mathcal{C} = \mathcal{X}$ is controlled invariant but not λ-contractive for any $\lambda \in [0,1)$. To see this, first note that, for every $x \in \mathcal{C}$, we have $A\,x + B\,u \in \mathcal{C}$ for the particular choice $u = 0 \in \mathcal{U}$, i.e., $\mathcal{C}$ is controlled invariant according to Def. 3. Now consider the point $x = (\,-5\;\;5\,)^T \in \mathcal{C}$ and note that (4) cannot be fulfilled since $A\,x + B\,u = (\,5 + u\;\;\; -5 + 2\,u\,)^T \notin \lambda\,\mathcal{C}$ for any $u \in [-1,1] = \mathcal{U}$

and any $\lambda \in [0,1)$. Thus, $\mathcal{C}$ is not λ-contractive for any $\lambda < 1$. However, we may compute λ-contractive subsets of $\mathcal{C}$ with the procedures stated in [3]. In Fig. 2, four tight outer approximations of the largest λ-contractive subsets of $\mathcal{C}$ for $\lambda \in \{0.8, 0.85, 0.9, 0.95\}$ are illustrated. The approximation corresponds to the 50th element of the sequence in [3, Eq. (3.1)] in all cases. Clearly, the λ-contractive subsets are harder to identify, more complex and smaller than the controlled invariant set $\mathcal{C}$.

In order to apply the control scheme presented in Prop. 1, we initially note that $\underline{r}_x = 5$, $\overline{r}_x = \sqrt{2} \cdot 5$ and $\overline{r}_u = 1$ satisfy the relations (13) and (14), where $\sigma_{\max}(S_2(A,B)) = 3$. We now choose $\mu = 0.99$ close to 1 and obtain $\tilde{\lambda} = 0.9986$ and $w > 707.1169$ from (15) and (16), respectively. Thus, $w = 708$ is a proper choice for the weight in (11). Note that (11) can be written as a linear program due to the polyhedral constraints $\mathcal{X}$ and $\mathcal{U}$ (see Rem. 1). Thus, the optimization problem (11) can be solved explicitly and the associated control law (17) is known to be piecewise affine. The resulting control law ϱ is illustrated in Fig. 2. The first two steps of the closed-loop trajectory emanating from $x_0 = (\,-3 \;\; 3\,)^T$ are illustrated in Fig. 1 (see the states connected by the solid lines). In fact, the trajectory of the controlled system is identical to the contractive trajectory already discussed in Sect. 3.1. It can be seen from Fig. 1 that the trajectory initially leaves the level set $\Psi_{\mathcal{C}}(x_0)\,\mathcal{C} = 0.6\,\mathcal{C}$ and returns to it in the second step. In fact, for the considered initial state x_0, it is impossible to steer the system to the interior of $\Psi_{\mathcal{C}}(x_0)\,\mathcal{C}$ without afore leaving $\Psi_{\mathcal{C}}(x_0)\,\mathcal{C}$.

The discussed example points out the crucial difference between classical λ-contractive controllers (see Sect. 2.2) and the new control strategy presented in Sect. 3. Basically, λ-contractive controllers try to minimize the distance to the origin in each time-step. Consequently, the trajectory of the controlled system will never leave the set $\Psi_{\mathcal{C}}(x_0)\,\mathcal{C}$. In contrast, the new controller associated with controlled invariant sets allows for short-time “excursions”, which are sometimes necessary to stabilize states within a controlled invariant set.

5 Conclusion and outlook

We proved a proposition on the existence of stabilizing control laws for linear constrained system on controlled invariant sets. More precisely, there exists, for any linear system (1) with constraints (2) and controlled invariant C-set $\mathcal{C}$, a stabilizing control law on $\mu\,\mathcal{C}$ for any $\mu \in (0,1)$. The proof is constructive. The resulting control law is defined by a convex receding horizon optimal control problem with horizon length equal to the system dimension n. The optimal control problem can be written as a multiparametric linear program if the sets $\mathcal{U}$ and $\mathcal{C}$ are polytopic.

Future work has to address the integration of the proposed controller into conventional MPC schemes. In fact, the presented approach allows to consider controlled invariant sets (which are not λ-contractive for any $\lambda < 1$) as terminal sets for MPC with guaranteed stability. A similar approach based on λ-contractive sets (with $\lambda < 1$) was recently published in [7].

References

[1] M. S. Bazaraa, H. D. Sherali, and C. M. Shetty. *Nonlinear Programming.* John Wiley, 3. ed., 2006.

[2] A. Bemporad, M. Morari, V. Dua, and E. N. Pistikopoulos. The explicit linear quadratic regulator for constrained systems. *Automatica*, 38(1):3–20, 2002.

[3] F. Blanchini. Ultimate boundedness control for uncertain discrete-time systems via set-induced Lyapunov functions. *IEEE Trans. Autom. Control*, 39(2):428–433, 1994.

[4] F. Blanchini. Set invariance in control. *Automatica*, 35:1747–1767, 1999.

[5] F. Blanchini and S. Miani. *Set-Theoretic Methods in Control.* Birkhäuser, 2008.

[6] S. Boyd and L. Vandenberghe. *Convex Optimization.* Cambrige University Press, 2004.

[7] S. Grammatico and G. Pannocchia. Achieving a large domain of attraction with short-horizon linear MPC via polyhedral Lyapunov functions. In *Proc. of 2013 European Control Conference*, pp. 1059–1064, 2013.

[8] P. O. Gutman and M. Cwikel. Admissible sets and feedback control for discrete time linear dynamic systems with bounded controls and states. *IEEE Trans. Autom. Control*, 31(4):373–376, 1986.

[9] M.L.J. Hautus. Stabilization, controllability and observability of linear autonomous systems. *Proc. of the Koninklijke Nederlandse Akademie van Wetenschappen*, 73(5):448–455, 1970.

[10] R. A. Horn and C. R. Johnson. *Matrix Analysis.* Cambrige University Press, 1985.

[11] C. N. Jones, M. Barić, and M. Morari. Multiparametric linear programming with applications to control. *European Journal of Control*, 13:152–170, 2007.

[12] R. E. Kalman. Contributions to the theory of optimal control. *Boletin de la Sociedad Mathematica Mexicana*, 5:102–119, 1960.

[13] S. S. Keerthi and E. G. Gilbert. Computation of minimum-time feedback control laws for discrete-time systems with state-control constraints. *IEEE Trans. Autom. Control*, 32(5):432–435, 1987.

[14] D. Q. Mayne, J. B. Rawlings, C.V. Rao, and P. O. M. Scokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 36:789–814, 2000.

[15] H. H. Rosenbrock. Distinctive problems in process control. *Chem. Engineering Progress*, 58:43–50, 1962.

[16] M. Schulze Darup. *Numerical Methods for the Investigation of Stabilizability of Constrained Systems.* PhD thesis, Ruhr-Universität Bochum, 2014.

[17] P. O. M. Scokaert and J. B. Rawlings. Constrained linear quadratic regulation. *IEEE Trans. Autom. Control*, 43(8):1163–1169, 1998.

[18] M. Vidyasagar. *Nonlinear System Analysis.* Society for Industrial Mathematics, 2002.