Preamble. This is a reprint of the article:

M. Schulze Darup and M. Mönnigmann. On general relations between null-controllable and controlled invariant sets for linear constrained systems. In *Proc. of the 53th IEEE Conference on Decision and Control*, pp. 6323–6328, 2014.

The digital object identifier (DOI) of the original article is:

`10.1109/CDC.2014.7040380`

On general relations between null-controllable and controlled invariant sets for linear constrained systems

Moritz Schulze Darup[†] *and* M. Mönnigmann[†]

Abstract

We prove some general relations between null-controllable and controlled invariant sets for linear systems with input and state constraints. We show that the closure of the largest null-controllable set is identical to the largest controlled invariant set. In order to prove this claim, we demonstrate that the interior of every controlled invariant set is null-controllable in the linear case. While some of these properties appear to be obvious, formal proofs are missing to the best of the authors' knowledge. To highlight the importance of careful proofs, we show that these properties are specific to linear systems and generally do not hold in the nonlinear case.

---

1 Introduction

We study null-controllable sets of linear discrete-time systems

$$x(k+1) = A\,x(k) + B\,u(k), \qquad x(0) = x_0, \tag{1}$$

with input and state constraints of the form

$$u(k) \in \mathcal{U} \subset \mathbb{R}^m, \qquad x(k) \in \mathcal{X} \subset \mathbb{R}^n, \qquad \forall\, k \in \mathbb{N}, \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ and where $\mathcal{U}$ and $\mathcal{X}$ are assumed to be convex and compact sets with the origin in their interiors. It is well-know that the *i-step null-controllable set* $\mathcal{N}_i$, i.e., the set of all states x_0 that can be steered to the origin in at most $i \in \mathbb{N}$ steps without violating the input and state constraints, can be computed from

$$\mathcal{N}_{i+1} = \mathcal{Q}(\mathcal{N}_i) \cap \mathcal{X} \quad \text{with} \quad \mathcal{N}_0 = \{0\}, \tag{3}$$

[†] M. Schulze Darup and M. Mönnigmann are with Automatic Control and Systems Theory, Department of Mechanical Engineering, Ruhr-Universität Bochum, 44801 Bochum, Germany. E-mail: `moritz.schulzedarup@rub.de`.

where

$$\mathcal{Q}(\mathcal{T}) := \{x \in \mathbb{R}^n \,|\, \exists u \in \mathcal{U} : A\,x + B\,u \in \mathcal{T}\} \tag{4}$$

refers to the so-called *one-step controllable set to* $\mathcal{T}$ (see, e.g., [6] or [9]). The sequence (3) is known to approach the *largest null-controllable set* (LNCS) $\mathcal{N}_{\max}$ (i.e., the set of all states x_0 that can be steered to the origin within a finite number of steps without violating the constraints) from inside, that is $\mathcal{N}_i \subseteq \mathcal{N}_{\max}$ for every $i \in \mathbb{N}$ [6]. An outer approximation of $\mathcal{N}_{\max}$ can be computed from[1]

$$\mathcal{C}_{i+1}^1 = \mathcal{Q}(\mathcal{C}_i^1) \cap \mathcal{X} \quad \text{with} \quad \mathcal{C}_0^1 = \mathcal{X}, \tag{5}$$

where $\mathcal{C}_i^1$ refers to the *i-step constraint-admissible set*, i.e., the set of all states x_0 that can be kept in $\mathcal{X}$ for at least i-steps without violating the constraints. The sequence (5) is known to approach the *largest controlled invariant set* (LCIS) $\mathcal{C}_{\max}^1$ from outside [1].

The main contribution of the paper is to *prove* that, in the linear case, the closure of the LNCS is identical to the LCIS. This result builds on the observation that the interior of every controlled invariant set is null-controllable. As a consequence, the elements of the sequences (3) and (5) become arbitrarily close for large $i \in \mathbb{N}$. The claimed characteristics appear self-evident, which may be the reason that formal proofs are missing (to the best of the authors' knowledge). We stress that careful proofs are important, since the listed characteristics are specific to linear systems and generally do not hold in the nonlinear case (see the motivating example in the beginning of Sect. 3).

The paper is organized as follows. We state notation and preliminaries in Sect. 2. The main result of the paper, i.e., the proof that $\mathrm{cl}(\mathcal{N}_{\max}) = \mathcal{C}_{\max}^1$ for linear systems, is given in Sect. 3. Finally, we illustrate our findings with two examples and state conclusions in Sects. 4 and 5, respectively.

2 Notation and Preliminaries

2.1 Notation

We denote matrices by capital letters, vectors and scalars by lowercase letters and sets by calligraphic letters. Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, $n, m \in \mathbb{N}$. Let $I_n \in \mathbb{R}^{n\times n}$ refer to the identity matrix. By B^T, $\mathrm{rk}(B)$, $\sigma_{\min}(B)$ and $\sigma_{\max}(B)$ denote the transpose, the rank and the smallest and largest singular value of the matrix B, respectively. Define

$$S_\nu(A,B) := (A^0B,\, A^1B, \ldots,\, A^{\nu-1}B),$$

for $\nu \in \mathbb{N}_+$, where $\mathbb{N}_+$ denotes positive integers.

Let $\mathcal{X}, \mathcal{T} \subseteq \mathbb{R}^n$ and $\mathbb{N}_{j,k} := \{i \in \mathbb{N} \,|\, j \leq i \leq k\}$. Let $\mathrm{int}(\mathcal{T})$ and $\mathrm{cl}(\mathcal{T})$ refer to the interior and the closure of the set $\mathcal{T}$, respectively. By $\mathcal{X} \oplus \mathcal{T} := \{x + t \,|\, x \in \mathcal{X},\, t \in \mathcal{T}\}$ denote the Minkowski sum of $\mathcal{X}$ and $\mathcal{T}$. Furthermore, let $\lambda \in \mathbb{R}$ and define the sets $\lambda\,\mathcal{X} := \{\lambda\,x \,|\, x \in \mathcal{X}\}$, $B\,\mathcal{X} := \{B\,x \,|\, x \in \mathcal{X}\}$ and $A^{-1}\,\mathcal{X} := \{x \,|\, A\,x \in \mathcal{X}\}$. Recall that $A^{-1}\,\mathcal{X}$ is well-defined even if A is not invertible. Note that we can write the one-step-set introduced in (4) as $\mathcal{Q}(\mathcal{T}) = A^{-1}(\mathcal{T} \oplus B\,(-\mathcal{U}))$ with the set operations introduced so far (see [6] for a discussion of this expression). Finally, by $\mathcal{B}^n(r) = \{x \in \mathbb{R}^n \,|\, \|x\|_2 \leq r\}$, denote a ball in $\mathbb{R}^n$ with radius $r \in \mathbb{R}_+$, where $\mathbb{R}_+$ denotes positive reals.

[1] Note that the superscript "1" in (5) is used to highlight the relation to expression (8) for the choice $\lambda = 1$.

2.2 Basic definitions and assumptions

A convex and compact set $\mathcal{T} \subset \mathbb{R}^n$ with $0 \in \text{int}(\mathcal{T})$ will be called C-set [2]. The so-called Minkowski function of a C-set $\mathcal{T}$ is defined by

$$\Psi_{\mathcal{T}}(x) := \inf\{\lambda \in \mathbb{R} \,|\, x \in \lambda\mathcal{T},\, \lambda \geq 0\}$$

(see [4, pp. 79-80] for details). Let ϑ_l be the shorthand notation for any control sequence $\vartheta_l = \{u(0), u(1), \ldots, u(l-1)\}$ of length $l \in \mathbb{N}_+$. The solution of (1) at time $k \in \mathbb{N}_{0,l}$ associated with a particular control sequence ϑ_l is denoted by $\varphi(k, x_0, \vartheta_l)$, where

$$\varphi(k, x_0, \vartheta_l) = A^k x_0 + \textstyle\sum_{j=0}^{k-1} A^{k-1-j} B\, u(j). \tag{6}$$

We call ϑ_l an *admissible control sequence*, if $u(k-1) \in \mathcal{U}$ and $\varphi(k, x_0, \vartheta_l) \in \mathcal{X}$ for every $k \in \mathbb{N}_{1,l}$. Let $\varrho : \mathbb{R}^n \to \mathcal{U}$ denote a control law. The solution of the *controlled system*

$$x(k+1) = A\, x(k) + B\, \varrho(x(k)), \qquad x(0) = x_0, \tag{7}$$

at time $k \in \mathbb{N}$ is denoted by $\varphi(k, x_0, \varrho)$. It will be clear from the context whether φ refers the solution of the uncontrolled (1) or controlled system (7).

We frequently use null-controllable, λ-contractive and controlled invariant sets.

Definition 1: *A set $\mathcal{T} \subseteq \mathcal{X}$ is called null-controllable if, for every $x_0 \in \mathcal{T}$, there exist an $l \in \mathbb{N}_+$ and an admissible control sequence ϑ_l such that $\varphi(l, x_0, \vartheta_l) = 0$. The largest null-controllable set is denoted by $\mathcal{N}_{\max}$.*

Definition 2: *Let $\lambda \in [0, 1]$. A $\mathcal{T} \subseteq \mathcal{X}$ is called λ-contractive if, for every $x_0 \in \mathcal{T}$, there exists a $u_0 \in \mathcal{U}$ such that $A\, x_0 + B\, u_0 \in \lambda\mathcal{T}$. For the special case $\lambda = 1$, a 1-contractive set $\mathcal{T}$ is also called controlled invariant. The largest λ-contractive set is denoted by $\mathcal{C}^\lambda_{\max}$.*

We make the following assumptions throughout the paper.

Assumption 1: *Let the pair (A, B) be controllable and let $\mathcal{X}$ and $\mathcal{U}$ be C-sets.*

Note that controllability of the pair (A, B) implies $\text{rk}(S_n(A, B)) = n$ [8, Def. 1].

2.3 Known Properties of Null-Controllable and Constraint-Admissible Sets

We summarize some well-known characteristics of null-controllable and constraint-admissible sets in the following. We first introduce a more general variant of (5), which reads

$$\mathcal{C}^\lambda_i = \mathcal{Q}(\lambda\, \mathcal{C}^\lambda_{i-1}) \cap \mathcal{X} \quad \text{with} \quad \mathcal{C}^\lambda_0 = \mathcal{X}, \tag{8}$$

where $\lambda \in [0, 1]$ is arbitrary but fixed (see [2] for further details). The sequence $\{\mathcal{C}^\lambda_i\}$ is known to approach the largest λ-contractive set $\mathcal{C}^\lambda_{\max}$ from outside [2]. Obviously, for $\lambda = 1$, Eq. (8) turns into the special case (5). We collect some important properties of the sequences (3) and (8) and their relations to $\mathcal{N}_{\max}$ and $\mathcal{C}^\lambda_{\max}$ in Tab. I.

Note that the elements of the sequences $\{\mathcal{N}_i\}$ and $\{\mathcal{C}^\lambda_i\}$ are C-sets (given $i \geq n$ [6, Lem. 4.3] and $0 \in \text{int}(\mathcal{C}^\lambda_{\max})$ [2, Thm. 3.1], respectively). The limit of a sequence[2] of C-sets need not be a C-set, however, and it is important to analyze the properties of the limits[2]

$$\mathcal{N}_\infty := \lim_{i\to\infty} \mathcal{N}_i \quad \text{and} \quad \mathcal{C}^\lambda_\infty := \lim_{i\to\infty} \mathcal{C}^\lambda_i$$

[2] A sequence of sets converges to a limit, if the limit superior and the limit inferior are equivalent [7, p. 21]. With regard to sequence (3), it is easy to show that the limit is such as in Tab. I, since $\liminf_{i\to\infty} \mathcal{N}_i = \bigcup_{i=0}^\infty \bigcap_{k=i}^\infty \mathcal{N}_k = \bigcup_{i=0}^\infty \mathcal{N}_i$ and $\limsup_{i\to\infty} \mathcal{N}_i = \bigcap_{i=0}^\infty \bigcup_{k=i}^\infty \mathcal{N}_k = \bigcup_{i=0}^\infty \mathcal{N}_i$ due to the identified monotonicity. Analogously, it can be shown that $\mathcal{C}^\lambda_\infty = \bigcap_{i=0}^\infty \mathcal{C}^\lambda_i$.

separately in this regard. The limit $\mathcal{C}_\infty^\lambda$ actually is a C-set (given $0 \in \operatorname{int}(\mathcal{C}_{\max}^\lambda)$) [2]. In particular $0 \in \operatorname{int}(\mathcal{C}_{\max}^\lambda)$ for $\lambda = 1$ since $0 \in \operatorname{int}(\mathcal{N}_{\max}) \subseteq \mathcal{C}_{\max}^1$. Consequently, since $\mathcal{C}_{\max}^\lambda = \mathcal{C}_\infty^\lambda$ according to [2], the LCIS is a C-set. In contrast, the limit of the sequence $\{\mathcal{N}_i\}$ may or may not be a C-set depending on the considered system. Thus, since $\mathcal{N}_{\max} = \mathcal{N}_\infty$ according to [6] (given Assum. 1 holds), the LNCS may or may not be a C-set. In fact, while the LNCS is always convex and bounded with $0 \in \operatorname{int}(\mathcal{N}_{\max})$, it may be open even though it is the limit of a sequence of closed sets. See Sect. 4 both for examples where $\mathcal{N}_{\max}$ is open and where it is closed. See [6, Ex. 4.2] or [4, p. 172] for further examples where $\mathcal{N}_{\max}$ is open and therefore not a C-set.

Table 1: Properties of the sequences (3) and (8) extracted from [1–6, 10].

	null-controllable sets	constraint-admissible sets
monotonicity	$\mathcal{N}_i \subseteq \mathcal{N}_{i+1} \subseteq \mathcal{N}_{\max}$	$\mathcal{C}_{\max}^\lambda \subseteq \mathcal{C}_{i+1}^\lambda \subseteq \mathcal{C}_i^\lambda$
elements	$\mathcal{N}_i$ is C-set if $i \geq n$	$\mathcal{C}_i^\lambda$ is C-set if $0 \in \operatorname{int}(\mathcal{C}_{\max}^\lambda)$
limit	$\mathcal{N}_\infty = \bigcup_{i=0}^\infty \mathcal{N}_i$	$\mathcal{C}_\infty^\lambda = \bigcap_{i=0}^\infty \mathcal{C}_i^\lambda$
relationships	$\mathcal{N}_{\max} = \mathcal{N}_\infty$, $\mathcal{N}_{\max} \subseteq \mathcal{C}_{\max}^1$	$\mathcal{C}_{\max}^\lambda = \mathcal{C}_\infty^\lambda$, $\mathcal{C}_{\max}^\lambda \subseteq \mathcal{N}_{\max}$ if $\lambda \in [0, 1)$

Since $\mathcal{N}_i$ and $\mathcal{C}_i^\lambda$ are C-sets (given $i \geq n$ and $0 \in \operatorname{int}(\mathcal{C}_{\max}^\lambda)$) and due to the convergence of (3) and (8) to $\mathcal{N}_{\max}$ respectively $\mathcal{C}_{\max}^\lambda$, for every $\epsilon > 0$, there exist $i, j \in \mathbb{N}$ such that

$$\mathcal{N}_{\max} \subseteq (1+\epsilon)\,\mathcal{N}_i \quad \text{and} \quad \mathcal{C}_j^\lambda \subseteq (1+\epsilon)\,\mathcal{C}_{\max}^\lambda, \tag{9}$$

respectively (see, e.g., [5] and [2]). Finally note that the LNCS is known to be controlled-invariant while every λ-contractive set (with $\lambda < 1$) is null-controllable (given Assum. 1 holds).

3 Important relations of null-controllable and controlled invariant sets

The main contribution of the paper is summarized in the following proposition.

Proposition 1: *Consider linear systems of the form* (1) *with constraints* (2) *and assume Assum. 1 holds. Then, the closure of the LNCS $\mathcal{N}_{\max}$ equals the LCIS $\mathcal{C}_{\max}^1$, i.e.,* $\operatorname{cl}(\mathcal{N}_{\max}) = \mathcal{C}_{\max}^1$.

While Prop. 1 seems natural, it is not self-evident. In fact, even for very simple nonlinear systems, the relation $\operatorname{cl}(\mathcal{N}_{\max}) = \mathcal{C}_{\max}^1$ does not hold. Consider for example the bilinear system

$$x(k+1) = 1.2\,x(k) + (0.4 + 0.8\,x(k))\,u(k)$$

with $\mathcal{X} = [-2, 2]$ and $\mathcal{U} = [-1, 1]$. It is easy to show that the LNCS[3] reads $\mathcal{N}_{\max} = (-0.4, 2.0]$ while the LCIS is $\mathcal{C}_{\max}^1 = [-2.0, 2.0]$ [11]. Clearly, $\operatorname{cl}(\mathcal{N}_{\max}) = [-0.4, 2.0] \neq \mathcal{C}_{\max}^1$.

The proof of Prop. 1 builds on the following lemma, which we prove in Sect. 3.1.

[3] Note that the definitions of null-controllable and controlled invariant sets for nonlinear systems are analogue to Defs. 1 and 2 except that the nonlinear system dynamics are considered.

Lemma 1: *The interior of the largest controlled invariant set $\mathcal{C}^1_{\max}$ is null-controllable.*

Proof of Prop. 1. We have
$$\operatorname{int}(\mathcal{C}^1_{\max}) \subseteq \mathcal{N}_{\max} \subseteq \mathcal{C}^1_{\max} \tag{10}$$
according to Lem. 1 and Tab. I, respectively. Evaluating the closure of (10) yields

$$\operatorname{cl}(\operatorname{int}(\mathcal{C}^1_{\max})) = \operatorname{cl}(\mathcal{C}^1_{\max}) \subseteq \operatorname{cl}(\mathcal{N}_{\max}) \subseteq \operatorname{cl}(\mathcal{C}^1_{\max}).$$

Taking into account that $\mathcal{C}^1_{\max}$ is closed (since it is a C-set, see Sect. 2.3), i.e., $\operatorname{cl}(\mathcal{C}^1_{\max}) = \mathcal{C}^1_{\max}$, we infer $\mathcal{C}^1_{\max} \subseteq \operatorname{cl}(\mathcal{N}_{\max}) \subseteq \mathcal{C}^1_{\max}$, which proves the claim. ■

As a direct consequence of Prop. 1 and Lem. 1, the elements of the sequences $\{\mathcal{N}_i\}$ and $\{\mathcal{C}^1_j\}$ become arbitrarily close for $i, j \to \infty$. This observation is formalized in the following corollary.

Corollary 1: *For every $\epsilon > 0$, there exist $i, j \in \mathbb{N}$ such that*

$$\mathcal{C}^1_j \subseteq (1+\epsilon)\,\mathcal{N}_i. \tag{11}$$

Proof. Set $\delta := \sqrt{1+\epsilon} - 1$ and note that $\delta > 0$. According to Eq. (9), there exist $i, j \in \mathbb{N}$ such that
$$\mathcal{N}_{\max} \subseteq (1+\delta)\,\mathcal{N}_i \quad \text{and} \quad \mathcal{C}^1_j \subseteq (1+\delta)\,\mathcal{C}^1_{\max}, \tag{12}$$
respectively. Evaluating the closure of the first relation in (12) yields $\operatorname{cl}(\mathcal{N}_{\max}) \subseteq (1+\delta)\operatorname{cl}(\mathcal{N}_i)$. We have $\operatorname{cl}(\mathcal{N}_{\max}) = \mathcal{C}^1_{\max}$ and $\operatorname{cl}(\mathcal{N}_i) = \mathcal{N}_i$ according to Prop. 1 and since $\mathcal{N}_i$ is closed (since it is a C-set, see Sect. 2.3). Thus, we find

$$\mathcal{C}^1_{\max} \subseteq (1+\delta)\,\mathcal{N}_i. \tag{13}$$

Multiplying (13) by $(1+\delta)$ and substituting the result into the second relation in (12) results in
$$\mathcal{C}^1_j \subseteq (1+\delta)\,\mathcal{C}^1_{\max} \subseteq (1+\delta)^2\,\mathcal{N}_i = (1+\epsilon)\,\mathcal{N}_i,$$
where the last relation holds by definition of δ. ■

We prove Lem. 1 in Sect. 3.1. Note that there exists a trivial proof of Lem. 1 (see the appendix), if the following familiar conjecture holds (cf. [4, pp. 168 ff.]).[4]

Conjecture 1: *For every $\epsilon > 0$, there exists a $\lambda \in [0, 1)$ such that*

$$\mathcal{C}^1_{\max} \subseteq (1+\epsilon)\,\mathcal{C}^\lambda_{\max}. \tag{14}$$

However, while Conj. 1 applies for many linear systems, a formal proof for the correctness is missing (and not straightforward) to the best of the authors' knowledge.

[4] We thank one of the anonymous reviewers for bringing to our attention that the proof of Lem. 1 would be trivial if Conj. 1 could be proven.

3.1 The null-controllable interior of controlled invariant sets

In this section, we show that the fundamental statement in Lem. 2 given below holds. Obviously, Lem. 2 immediately proves Lem. 1 since $\mathcal{C}^1_{\max}$ is a C-set according to Sect. 2.3 and controlled invariant by definition (see Def. 2).

Lemma 2: *Let $\mathcal{T} \subseteq \mathcal{X}$ be a controlled invariant C-set. Then,* $\mathrm{int}(\mathcal{T})$ *is null-controllable.*

We prove Lem. 2 in the remainder of the section. Lemma 3 is needed as a preparation. Loosely speaking, Lem. 3 states that, for every initial state x_0 in the interior of a controlled invariant C-set $\mathcal{T}$, there exists a control sequence ϑ_n such that the trajectory has moved closer to the origin than x_0 after at most n steps (where n refers to the system dimension). We use the Minkowski function value to measure closeness to the origin. See Fig. 1 and the example in Sect. 4.2 for an illustration of the statement in Lem. 3. Basically, the lemma makes use of the fact that an unconstrained linear system can be steered to every state within n steps.

Lemma 3: *Let $\mathcal{T} \subseteq \mathcal{X}$ be a controlled invariant C-set and let $\mu \in (0,1)$ be arbitrary. Let $\underline{r}_u, \underline{r}_x, \overline{r}_x \in \mathbb{R}_+$ be such that $\mathcal{B}^m(\underline{r}_u) \subseteq \mathcal{U}$, $\mathcal{B}^n(\underline{r}_x) \subseteq \mathcal{T} \subseteq \mathcal{B}^n(\overline{r}_x)$ and*

$$\underline{r}_x \geq \sqrt{n}\, \sigma_{\max}(S_n(A,B))\, \underline{r}_u. \tag{15}$$

Then, for every $x_0 \in \mu\,\mathcal{T}$ there exist an admissible control sequence ϑ_n such that

$$\Psi_{\mathcal{T}}(\varphi(n, x_0, \vartheta_n)) \leq \tilde{\lambda}\, \Psi_{\mathcal{T}}(x_0), \tag{16}$$

where $\tilde{\lambda} \in [0,1)$ is defined as

$$\tilde{\lambda} := \max\left\{0, 1 - \sigma_{\min}(S_n(A,B))\, \frac{\underline{r}_u}{\overline{r}_x} \left(\frac{1}{\mu} - 1\right)\right\}. \tag{17}$$

Proof. First note that $\tilde{\lambda} < 1$, since $\underline{r}_u, \overline{r}_x \in \mathbb{R}_+$, since $\frac{1}{\mu} - 1 > 0$ for every $\mu \in (0,1)$, and since Assum. 1 implies $\sigma_{\min}(S_n(A,B)) > 0$. Further note that the choice of $\underline{r}_x$ and $\overline{r}_x$ implies

$$\overline{r}_x^{-1} \|x\|_2 \leq \Psi_{\mathcal{T}}(x) \leq \underline{r}_x^{-1} \|x\|_2 \tag{18}$$

for every $x \in \mathbb{R}^n$. Now, since $\mathcal{T}$ is controlled invariant there exists, for every $x_0 \in \partial\mathcal{T}$, a $u_0 \in \mathcal{U}$ such that $A\,x_0 + B\,u_0 \in \mathcal{T}$. Thus, there exists a function $\varrho_\partial : \partial\mathcal{T} \to \mathcal{U}$ such that $\varphi(1, x_0, \varrho_\partial) \in \mathcal{T}$, which implies $\Psi_{\mathcal{T}}(\varphi(1, x_0, \varrho_\partial)) \leq 1$. Based on ϱ_∂, we introduce the control law $\varrho : \mathcal{T} \to \mathcal{U}$ (see [3, p. 1753]), where

$$\varrho(x) := \begin{cases} \Psi_{\mathcal{T}}(x)\, \varrho_\partial\left(\frac{x}{\Psi_{\mathcal{T}}(x)}\right) & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0. \end{cases} \tag{19}$$

Note that $\frac{x}{\Psi_{\mathcal{T}}(x)} \in \partial\mathcal{T}$ for every $x \neq 0$. Moreover, it is easy to show that

$$\Psi_{\mathcal{T}}(\varphi(k, x_0, \varrho)) \leq \Psi_{\mathcal{T}}(x_0) \tag{20}$$

for every $x_0 \in \mathcal{T}$ and every $k \in \mathbb{N}$. Thus, for every $x_0 \in \mathcal{T}$, the admissible control sequence $\vartheta_n = \{\varrho(\varphi(0, x_0, \varrho)), \dots, \varrho(\varphi(n-1, x_0, \varrho))\}$ satisfies (16) for $\tilde{\lambda} = 1$. In order to show that (16) holds for $\tilde{\lambda} < 1$ as in (17), we study small perturbations of ϑ_n, i.e.,

$$u(k) = \varrho(\varphi(k, x_0, \varrho)) + \Delta u(k), \tag{21}$$

for every $k \in \mathbb{N}_{0,n-1}$. Since we have to take the input constraints into account, the perturbations $\Delta u(k)$ are subject to constraints. In order to specify constraints for $\Delta u(k)$, first note that we have $\Psi_{\mathcal{T}}(x_0) \leq \mu$ for every $x_0 \in \mu\,\mathcal{T}$. From the definition of ϱ in (19), we infer $\varrho(x_0) \in \Psi_{\mathcal{T}}(x_0)\,\mathcal{U} \subseteq \mu\,\mathcal{U}$ for every $x_0 \in \mu\,\mathcal{T}$. Moreover, in combination with (20), we find

$$\varphi(k, x_0, \varrho) \in \mu\,\mathcal{T} \quad \text{and} \quad \varrho(\varphi(k, x_0, \varrho)) \in \mu\,\mathcal{U} \tag{22}$$

for every $x_0 \in \mu\,\mathcal{T}$ and every $k \in \mathbb{N}$. For the following, we note that the choices $\Delta\overline{r}_x := (1-\mu)\,\underline{r}_x$ and $\Delta\overline{r}_u := (1-\mu)\,\underline{r}_u$ are such that

$$\mu\,\mathcal{T} \oplus \mathcal{B}^n(\Delta\overline{r}_x) \subseteq \mathcal{T} \quad \text{and} \quad \mu\,\mathcal{U} \oplus \mathcal{B}^m(\Delta\overline{r}_u) \subseteq \mathcal{U}, \tag{23}$$

respectively. To see this, note that, for every $\Delta x \in \mathcal{B}^n(\Delta\overline{r}_x)$, we have $\Psi_{\mathcal{T}}(\Delta x) \leq \frac{\|\Delta x\|_2}{\underline{r}_x} \leq \frac{\Delta\overline{r}_x}{\underline{r}_x} = 1-\mu$ according to (18) and by definition of $\Delta\overline{r}_x$, respectively. Thus, the first (and analogously the second) relation in (23) holds since

$$\Psi_{\mathcal{T}}(x+\Delta x) \leq \Psi_{\mathcal{T}}(x) + \Psi_{\mathcal{T}}(\Delta x) \leq \mu + 1 - \mu = 1 \tag{24}$$

for every $x \in \mu\,\mathcal{T}$ and every $\Delta x \in \mathcal{B}^n(\Delta\overline{r}_x)$, where the first inequality in (24) applies according to [4, Prop. 3.12]. In combination with (22), the relations (23) can be used as follows. Obviously, for any trajectory with initial condition $x_0 \in \mu\,\mathcal{T}$, the modified inputs (21) will not violate the constraints $\mathcal{U}$, if the perturbations satisfy $\|\Delta u(k)\|_2 \leq \Delta\overline{r}_u$ for every $k \in \mathbb{N}$. The perturbed inputs result in a perturbed trajectory, which must respect the state constraints. The perturbed states can be written as

$$x(k) = \varphi(k, x_0, \varrho) + \Delta x(k), \tag{25}$$

where $\Delta x(k) = \sum_{j=0}^{k-1} A^{k-1-j} B\,\Delta u(j)$ for all $k \in \mathbb{N}_{0,n}$ (cf., Eq. (6)). Obviously, $\Delta x(0) = 0$ and

$$\Delta x(k) = S_k(A,B) \begin{pmatrix} \Delta u(k-1) \\ \vdots \\ \Delta u(0) \end{pmatrix} \tag{26}$$

for all $k \in \mathbb{N}_{1,n}$. Now, initially choose any $\Delta r_u \in (0, \Delta\overline{r}_u]$ and assume $\|\Delta u(k)\|_2 \leq \Delta r_u$ for all $k \in \mathbb{N}$. Then, according to (26), we have

$$\begin{aligned} \|\Delta x(k)\|_2 &\leq \|S_k(A,B)\|_2 \sqrt{\sum_{j=0}^{k-1} \|\Delta u(j)\|_2^2}, \\ &\leq \sigma_{\max}(S_k(A,B))\,\sqrt{k}\,\Delta r_u \end{aligned} \tag{27}$$

for every $k \in \mathbb{N}_+$. It is easy to prove that the relation $\sigma_{\max}(S_k(A,B)) \leq \sigma_{\max}(S_{k+1}(A,B))$ holds for all $k \in \mathbb{N}_+$. Hence, introducing

$$\Delta r_x := \sigma_{\max}(S_n(A,B))\,\sqrt{n}\,\Delta r_u \tag{28}$$

yields $\|\Delta x(k)\|_2 \leq \Delta r_x$ for all $k \in \mathbb{N}_{0,n}$. In the following, we choose $\Delta r_u \leq \Delta\overline{r}_u$ such that $\Delta r_x \leq \Delta\overline{r}_x$, where Δr_x is defined as in (28). Thus, any choice of the perturbations $\Delta u(k)$, $k \in \mathbb{N}_{0,n-1}$, such that $\|\Delta u(k)\|_2 \leq \Delta r_u$ results in a trajectory that is guaranteed to satisfy the constraints (2) for the first n steps. Obviously, any choice $\Delta r_u \in \mathbb{R}_+$ with

$$\Delta r_u \leq \min\left\{\Delta\overline{r}_u, \frac{\Delta\overline{r}_x}{\sqrt{n}\,\sigma_{\max}(S_n(A,B))}\right\} \tag{29}$$

satisfies these conditions, where relation (28) was used to link Δr_x to Δr_u. An appropriate choice that respects (29) is

$$\Delta r_u := (1-\mu)\,\underline{r}_u = \Delta\overline{r}_u. \tag{30}$$

To see this, note that $\min\left\{\Delta\overline{r}_u, \frac{\Delta\overline{r}_x}{\sqrt{n}\,\sigma_{\max}(S_n(A,B))}\right\}$

$$= (1-\mu)\,\min\left\{\underline{r}_u, \frac{\underline{r}_x}{\sqrt{n}\,\sigma_{\max}(S_n(A,B))}\right\} \geq (1-\mu)\,\underline{r}_u,$$

where the two relations hold by definitions of $\Delta\overline{r}_u$ and $\Delta\overline{r}_x$ (stated below (22)) and according to (15), respectively.

Up to now, we showed choosing $\Delta u(k) \in \mathcal{B}^m(\Delta r_u)$ implies the constraints (2) are fulfilled for all $k \in \mathbb{N}_{0,n}$. It remains to show that there exists an admissible perturbed input sequence (21) that steers the system closer to the origin than x_0. To see this first note that, for every $\Delta x(n) \in \mathbb{R}^n$, there exist n inputs $\Delta u(0), \ldots, \Delta u(n-1)$ such that $\Delta x(n) = \sum_{j=0}^{n-1} A^{k-1-j} B\,\Delta u(j)$. In fact, since $S_n(A,B)$ has full rank according to Assum. 1 ((A,B) controllable implies $\mathrm{rk}(S_n(A,B)) = n$), the n inputs can be computed from

$$\begin{pmatrix} \Delta u(n-1) \\ \vdots \\ \Delta u(0) \end{pmatrix} = S_n(A,B)^{-1}\Delta x(n), \tag{31}$$

where $S_n(A,B)^{-1}$ refers to the Moore–Penrose pseudoinverse. Now let

$$\Delta r_n := \sigma_{\min}(S_n(A,B))\,\Delta r_u \tag{32}$$

and not that $\Delta r_n \leq \Delta r_x$. Then, for every $\Delta x(n) \in \mathcal{B}^n(\Delta r_n)$ the required inputs $\Delta u(0), \ldots, \Delta u(n-1)$ that steer the system to $\Delta x(n)$ satisfy $\|\Delta u(k)\|_2 \leq r_6$ for all $k \in \mathbb{N}_{0,n-1}$. This follows from

$$\begin{aligned} \|S_n(A,B)^{-1}\Delta x(n)\|_2 &\leq \|S_n(A,B)^{-1}\|_2\,\|\Delta x(n)\|_2, \\ &\leq \tfrac{1}{\sigma_{\min}(S_n(A,B))}\,\Delta r_n = \Delta r_u, \end{aligned}$$

where the particular relations hold due to sub-multiplicativity and $\|\Delta x(n)\|_2 \leq \Delta r_n$, since $S_n(A,B)$ has full rank and by definition of Δr_n, respectively. Combining (31) and $\|S_n(A,B)^{-1}\Delta x(n)\|_2 \leq \Delta r_u$ yields

$$\left\| \begin{pmatrix} \Delta u(n-1) \\ \vdots \\ \Delta u(0) \end{pmatrix} \right\|_2 \leq \Delta r_u$$

which implies $\|\Delta u(k)\|_2 \leq \Delta r_u$ for every $k \in \mathbb{N}_{0,n-1}$. In other words, for every $x_0 \in \mu\,\mathcal{T}$ and every $x(n) \in \varphi(n, x_0, \varrho) \oplus \mathcal{B}^n(\Delta r_n)$, there exists an admissible control sequence $\vartheta_n = \{u(0), \ldots, u(n-1)\}$, such that $x(n) = \varphi(n, x_0, \vartheta_n)$. In order to minimize the Euclidean distance to the origin, we may choose

$$x(n) = \hat{\lambda}\,\varphi(n, x_0, \varrho), \tag{33}$$

i.e., $\Delta x(n) = (\hat{\lambda} - 1)\,\varphi(n, x_0, \varrho)$, where $\hat{\lambda} \in [0,1)$ is defined by

$$\hat{\lambda} := \begin{cases} 0 & \text{if} \quad \|\varphi(n, x_0, \varrho)\|_2 \leq \Delta r_n, \\ 1 - \frac{\Delta r_n}{\|\varphi(n, x_0, \varrho)\|_2} & \text{if} \quad \|\varphi(n, x_0, \varrho)\|_2 > \Delta r_n. \end{cases} \tag{34}$$

Note that (33) fulfills $\|\Delta x(n)\|_2 \leq \Delta r_n$, since

$$\|(\hat{\lambda}-1)\,\varphi(n,x_0,\varrho)\|_2 = |\hat{\lambda}-1|\,\|\varphi(n,x_0,\varrho)\|_2 \\ = \begin{cases} \|\varphi_{\hat{\varrho}}(n,x_0)\|_2 & \text{if} \quad \|\varphi(n,x_0,\varrho)\|_2 \leq \Delta r_n, \\ \Delta r_n & \text{if} \quad \|\varphi(n,x_0,\varrho)\|_2 > \Delta r_n. \end{cases}$$

Choosing $x(n)$ as in (33) and computing the associated admissible control sequence ϑ_n such that $\varphi(n,x_0,\vartheta_n) = x(n)$ according to (31) and (21) results in

$$\Psi_{\mathcal{T}}(\varphi(n,x_0,\vartheta_n)) = \hat{\lambda}\,\Psi_{\mathcal{T}}(\varphi(n,x_0,\varrho)) \leq \hat{\lambda}\,\Psi_{\mathcal{T}}(x_0), \tag{35}$$

where the first and the second relation hold due to [4, Prop. 3.12] and (20), respectively.

With regard to (16), it remains to show that $\hat{\lambda}$ as in (34) is smaller than or equal to $\tilde{\lambda}$ as in (17). In case of $\hat{\lambda}=0$, $\hat{\lambda} \leq \tilde{\lambda}$ obviously holds since $\tilde{\lambda} \geq 0$. It remains to study the second case in (34), where $\|\varphi(n,x_0,\varrho)\|_2 > \Delta r_n$. With regard to (17), we have $\hat{\lambda} \leq \tilde{\lambda}$, if

$$\frac{\Delta r_n}{\|\varphi(n,x_0,\varrho)\|_2} \geq \sigma_{\min}(S_n(A,B))\,\frac{\underline{r}_u}{\overline{r}_x}\left(\frac{1}{\mu}-1\right). \tag{36}$$

Relation (36) holds since

$$\begin{aligned} \tfrac{\Delta r_n}{\|\varphi(n,x_0,\varrho)\|_2} &= \sigma_{\min}(S_n(A,B))\,\tfrac{\underline{r}_u(1-\mu)}{\|\varphi(n,x_0,\varrho)\|_2} \\ &\geq \sigma_{\min}(S_n(A,B))\,\tfrac{\underline{r}_u(1-\mu)}{\overline{r}_x\Psi_{\mathcal{T}}(\varphi(n,x_0,\varrho))} \\ &\geq \sigma_{\min}(S_n(A,B))\,\tfrac{\underline{r}_u}{\overline{r}_x}\,\tfrac{1-\mu}{\Psi_{\mathcal{T}}(x_0)} \\ &\geq \sigma_{\min}(S_n(A,B))\,\tfrac{\underline{r}_u}{\overline{r}_x}\,\tfrac{1-\mu}{\mu}, \end{aligned}$$

where the first relation holds by definition of Δr_n and Δr_u in (32) and (30). The second and the third relation follow from Eqs. (18) and (20), respectively. Finally, the last relation holds due to $x_0 \in \mu\,\mathcal{T}$, i.e., $\Psi_{\mathcal{T}}(x_0) \leq \mu$. ■

Finally, Lemma 2 can be shown with Lem. 3.

Proof of Lem. 2. Let $\underline{r}_u, \underline{r}_x, \overline{r}_x \in \mathbb{R}_+$ be as in Lem. 3 and let $x_0 \in \mathrm{int}(\mathcal{T})$ be arbitrary but fixed. Set $\mu_1 := \Psi_{\mathcal{T}}(x_0)$ and note that $\mu_1 < 1$. Thus, according to Lem. 3, there exists an admissible control sequence ϑ_n such that

$$\Psi_{\mathcal{T}}(\varphi(n,x_0,\vartheta_n)) \leq \tilde{\lambda}_1\,\Psi_{\mathcal{T}}(x_0), \tag{37}$$

where $\tilde{\lambda}_1 < 1$ refers to $\tilde{\lambda}$ as in (17) with $\mu = \mu_1$. Set $\mu_2 = \Psi_{\mathcal{T}}(\varphi(n,x_0,\vartheta_n))$ and note that $\mu_2 \leq \tilde{\lambda}_1\,\mu_1 < 1$. Hence, there exists another admissible control sequence ϑ_n such that

$$\Psi_{\mathcal{T}}(\varphi(n,\varphi(n,x_0,\vartheta_n),\vartheta_n)) \leq \tilde{\lambda}_2\,\Psi_{\mathcal{T}}(\varphi(n,x_0,\vartheta_n)), \tag{38}$$

where $\tilde{\lambda}_2 < 1$ refers to $\tilde{\lambda}$ as in (17) with $\mu = \mu_2$. Concatenating the two control sequences results in an admissible control sequence of length $2n$. Both (37) and (38) are fulfilled for this control sequence. Moreover, since $\mu_2 \leq \mu_1$, we obtain $\tilde{\lambda}_2 \leq \tilde{\lambda}_1$ from (17). In general, let $l \in \mathbb{N}_+$, then there exist $\tilde{\lambda}_1,\dots,\tilde{\lambda}_l \in [0,1)$ with $\tilde{\lambda}_i \leq \tilde{\lambda}_1$ and an admissible control sequence ϑ_{ln} such that

$$\Psi_{\mathcal{T}}(\varphi(i\,n,x_0,\vartheta_{ln})) \leq \tilde{\lambda}_i\,\Psi_{\mathcal{T}}(\varphi((i-1)\,n,x_0,\vartheta_{ln})) \tag{39}$$

for every $i \in \mathbb{N}_{1,l}$. From (39), we infer

$$\Psi_{\mathcal{T}}(\varphi(l\,n,x_0,\vartheta_{ln})) \leq \mu_1 \textstyle\prod_{i=1}^{l} \tilde{\lambda}_i \leq \mu_1\,\tilde{\lambda}_1^l. \tag{40}$$

which implies $\varphi(l\,n, x_0, \vartheta_{ln}) \in \mu_1 \tilde{\lambda}_1^l \mathcal{T}$. Since $0 \in \mathrm{int}(\mathcal{N}_n)$ (since it is a C-set, see Sect. 2.3) and since $\tilde{\lambda}_1 \in [0,1)$, for every $x_0 \in \mathrm{int}(\mathcal{T})$ there exists a $l \in \mathbb{N}_+$ such that $\mu_1 \tilde{\lambda}_1^l \mathcal{T} \subseteq \mathcal{N}_n$. Thus, x_0 can be steered to $\mathcal{N}_n$ in at most $l\,n$ steps without violating the input and state constraints. Since any state in $\mathcal{N}_n$ can be steered to the origin in at most n steps, there exists an admissible control sequence $\vartheta_{l(n+1)}$ of length $l\,(n+1)$ such that $\varphi(l\,(n+1), x_0, \vartheta_{l(n+1)}) = 0$. Since $x_0 \in \mathrm{int}(\mathcal{T})$ was arbitrary, $\mathrm{int}(\mathcal{T})$ is null-controllable. ■

4 Examples

We analyze two examples to illustrate the findings in Sect. 3. In order to facilitate the numerical evaluation of sets $\mathcal{N}_i$ and $\mathcal{C}_i$, we consider polytopic C-sets $\mathcal{X}$ and $\mathcal{U}$ in the examples. However, it is important to note that the statements in Sect. 3 hold for arbitrary C-sets $\mathcal{X}$ and $\mathcal{U}$.

4.1 Example 1: Illustration of Prop. 1 and Cor. 1

Consider system (1) with $A = 1.2$ and $B = 1$ and constraints $\mathcal{X} = [-10, 10]$ and $\mathcal{U} = [-1, 1]$. It is easy to show that the null-controllable sets $\mathcal{N}_i$ and the constraint-admissible sets $\mathcal{C}_j^1$ are

$$\mathcal{N}_i = [-\nu_i, \nu_i] \quad \text{and} \quad \mathcal{C}_j^1 = [-\kappa_j, \kappa_j], \tag{41}$$

where

$$\nu_i = 5\left(1 - \frac{1}{1.2^i}\right) \quad \text{and} \quad \kappa_j = 5\left(1 + \frac{1}{1.2^j}\right).$$

Obviously, the sequence ν_i approaches $\kappa_\infty := 5$ from below while κ_j approaches κ_∞ from above. The LCIS is given by the closed set $\mathcal{C}_{\max}^1 = [-5, 5]$. In contrast, the LNCS is open. In fact, $\mathcal{N}_{\max} = \bigcup_{i=0}^{\infty} \mathcal{N}_i = (-5, 5)$. Note that $x_0 = 5$ (as well as $x_0 = -5$) is not an element of $\mathcal{N}_{\max}$, since x_0 cannot be steered closer to the origin due to $A\,x_0 + B\,u_0 = 6 + u_0 \geq 5$ for every $u_0 \in \mathcal{U}$. However, the claim in Prop. 1 holds since $\mathrm{cl}(\mathcal{N}_{\max}) = [-5, 5] = \mathcal{C}_{\max}^1$. Moreover, the elements of the sequences $\{\mathcal{N}_i\}$ and $\{\mathcal{C}_j^1\}$ become arbitrarily close as predicted by Cor. 1. To see this, note that condition (11) reads

$$\kappa_j = 5\left(1 + \frac{1}{1.2^j}\right) \leq (1+\epsilon)\,\nu_i = 5\,(1+\epsilon)\left(1 - \frac{1}{1.2^i}\right) \tag{42}$$

for this example. Relation (42) holds for $i = j = \left\lceil \frac{\ln\left(1+\frac{2}{\epsilon}\right)}{\ln(1.2)} \right\rceil$.

4.2 Example 2: Illustration of Lem. 3

Consider system (1) with

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and constraints

$$\mathcal{X} = \{x \in \mathbb{R}^2 \,|\, |x_1| \leq 5,\, |x_2| \leq 5\} \quad \text{and} \quad \mathcal{U} = [-1, 1].$$

It is easy to show that the sets $\mathcal{N}_i$ and $\mathcal{C}_i^1$ evaluate to

$$\mathcal{N}_i = \left\{x \in \mathbb{R}^2 \,\middle|\, |x_1| \leq \min\{\lceil \tfrac{i}{2} \rceil, 5\},\, |x_2| \leq \min\{\lfloor \tfrac{i}{2} \rfloor, 5\}\right\}$$

and $\mathcal{C}_i^1 = \mathcal{X}$ for every $i \in \mathbb{N}$, respectively. We consequently find $\mathcal{C}_{\max} = \mathcal{X}$ and $\mathcal{N}_{\max} = \mathcal{N}_{10} = \mathcal{X}$ in accordance with Prop. 1. Thus, $\mathcal{N}_{\max}$ is closed for this example and $\mathrm{cl}(\mathcal{N}_{\max}) = \mathcal{N}_{\max} = \mathcal{C}_{\max}^1$. Moreover, for the choice $i = 10$ and $j = 0$, condition (11) is trivially fulfilled for every $\epsilon > 0$.

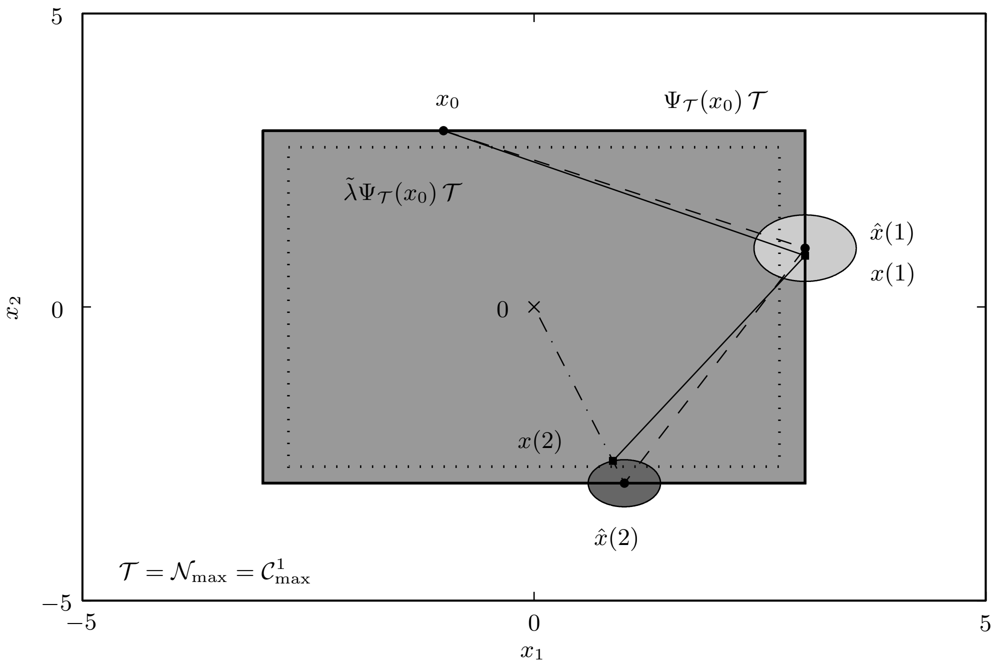

Figure 1: Illustration of two trajectories (of Ex. 2) that are instrumental to prove Lem. 2. The dashed trajectory along the states $\hat{x}(k) = \varphi(k, x_0, \varrho)$, $k \in \mathbb{N}_{0,2}$, refers to the solution of the controlled system (7) based on the auxiliary control law $\varrho(x) = 0$. Obviously, $\varphi(k, x_0, \varrho)$ remains on the boundary of the set $\Psi_{\mathcal{T}}(x_0)\,\mathcal{T}$ for the present example. In contrast, the solid trajectory via $x(k) = \varphi(k, x_0, \vartheta_2)$ enters $\tilde{\lambda}\Psi_{\mathcal{T}}(x_0)\,\mathcal{T}$ (dotted rectangle) in the second step. In fact, within two steps, the system (1) can be steered to any state in the smaller (dark gray) circle $\varphi(2, x_0, \varrho) + \mathcal{B}^2(\Delta r_n)$ without leaving the larger (light gray) circle $\varphi(1, x_0, \varrho) + \mathcal{B}^2(\Delta r_x)$ in the first step and without violating the input constraints.

It is important to note that the set $\mathcal{T} = \mathcal{N}_\infty = \mathcal{C}_{\max}^1$ (and in addition every set $\mathcal{N}_i$ with $i \geq 2$) is controlled invariant but not λ-contractive for any $\lambda \in [0, 1)$. To see this, consider for example the point $x_0 = (-4 \;\; 5)^T \in \mathcal{T}$ and note that $A\,x_0 + B\,u_0 = (5 \;\; 4 + u_0)^T \notin \lambda\,\mathcal{T}$ for every $u_0 \in \mathcal{U}$ and every $\lambda \in [0, 1)$. Thus, in accordance with Def. 2, $\mathcal{T}$ cannot be λ-contractive for any $\lambda \in [0, 1)$. In other words, there exist many $x_0 \in \mathcal{T}$ that cannot be steered closer to the origin within one time step (where closeness to the origin is measured by the Minkowski function value). In particular, it is easy to verify that this claim holds for every $x_0 \in \mathcal{T}$ with $|(x_0)_1| \leq |(x_0)_2|$.

However, according to Lem. 2, we are able to steer every $x_0 \in \mathrm{int}(\mathcal{T})$ to the origin. Consider for example the point $x_0 = (-1 \;\; 3)^T \in \mathrm{int}(\mathcal{T})$ and note that $\underline{r}_u = 1$, $\underline{r}_x = 5$ and $\overline{r}_x = 5\sqrt{2}$ fulfill the assumptions in Lem. 3. Obviously, $\Psi_{\mathcal{T}}(x_0) = 0.6$, i.e, $x_0 \in 0.6\,\mathcal{T}$. Consequently, choose $\mu = 0.6$. According to Lem. 3, for every $x_0 \in \mu\,\mathcal{T}$, there exists an admissible control sequence ϑ_2 such that $\Psi_{\mathcal{T}}(\varphi(2, x_0, \vartheta_2)) \leq \tilde{\lambda}\,\Psi_{\mathcal{T}}(x_0)$, where $\tilde{\lambda} = 1 - \frac{1}{5\sqrt{2}}\left(\frac{1}{\mu} - 1\right) = 1 - \frac{\sqrt{2}}{15} \approx 0.906$ follows from (17). A suitable trajectory, namely the one that is instrumental for the proof of Lem. 3, is illustrated in Fig. 1 (solid line). Following the proof with regard to the present example, we find that the control law $\varrho : \mathbb{R}^n \to \mathcal{U}$ with $\varrho(x) = 0$ for every $x \in \mathbb{R}^n$ is such that condition (20) holds. Moreover, with $\Delta r_u = \Delta \overline{r}_u =$

$(1-\mu)\,\underline{r}_u = 0.4$, $\Delta r_x = 0.4\sqrt{2}$ and $\Delta r_n = 0.4$, we find that x_0 can be steered to any state in the ball $\varphi(2, x_0, \varrho) + \mathcal{B}^2(\Delta r_n)$, where $\varphi(2, x_0, \varrho) = (1 \;\; -3)^T$ (see Fig. 1 for an illustration of this region). Choosing $x(n) = \hat{\lambda}\,\varphi(2, x_0, \varrho)$ as in (33) with $\hat{\lambda} = 1 - \frac{0.4}{\sqrt{10}} \approx 0.874$ according to (34) results in $\varphi(2, x_0, \vartheta_2) = \hat{\lambda}\,\varphi(2, x_0, \varrho) = (0.874 \;\; -2.621)^T$, where the admissible control sequence $\vartheta_2 = \{u(0), u(1)\} = \{-0.126\,,\, 0.379\}$ follows from (31) in combination with (21) and (25). Obviously, $\varphi(2, x_0, \vartheta_2)$ is closer to the origin than x_0. In fact, $\Psi_{\mathcal{T}}(\varphi(2, x_0, \vartheta_2)) = \frac{2.621}{5} \leq \tilde{\lambda}\,\Psi_{\mathcal{T}}(x_0) = 0.906 \cdot 0.6 = \frac{2.718}{5}$ as predicted by Lem. 1.

5 Conclusion and outlook

We presented formal proofs for some important relations between null-controllable and controlled invariant sets for linear constrained systems. The main result of the paper was to show that the closure of the largest null-controllable set $\mathcal{N}_{\max}$ is always equivalent to the largest controlled invariant set $\mathcal{C}^1_{\max}$ (given Assumption 1 holds). We highlighted that this results is specific to linear systems and generally does not hold in the nonlinear case.

Future work has to address the extension to stabilizable sets. Moreover, some results of this paper (more precisely Lem. 3) can be used to design simple null-controlling as well as stabilizing feedback laws (see [12] for details).

Acknowledgment

This research was partly funded by Deutsche Forschungsgemeinschaft (MO 1086/11-1).

References

[1] D. P. Bertsekas. Infinite-time reachability of state-space regions by using feedback control. *IEEE Trans. Autom. Control*, 17:604–613, 1972.

[2] F. Blanchini. Ultimate boundedness control for uncertain discrete-time systems via set-induced Lyapunov functions. *IEEE Trans. Autom. Control*, 39(2):428–433, 1994.

[3] F. Blanchini. Set invariance in control. *Automatica*, 35:1747–1767, 1999.

[4] F. Blanchini and S. Miani. *Set-Theoretic Methods in Control*. Birkhäuser, 2008.

[5] M. Cwikel and P. O. Gutman. Convergence of an algorithm to find maximal state constraint sets for discrete-time linear dynamical systems with bounded controls and states. *IEEE Trans. Autom. Control*, 31(5):457–459, 1986.

[6] P. O. Gutman and M. Cwikel. An algorithm to find maximal state constraint sets for discrete-time linear dynamical systems with bounded control and states. *IEEE Trans. Autom. Control*, 32(3):251–253, 1987.

[7] F. Hausdorff. *Set theory*. Chelsea, 1957.

[8] M.L.J. Hautus. Stabilization, controllability and observability of linear autonomous systems. *Proc. of the Koninklijke Nederlandse Akademie van Wetenschappen*, 73(5):448–455, 1970.

[9] S. S. Keerthi and E. G. Gilbert. Computation of minimum-time feedback control laws for discrete-time systems with state-control constraints. *IEEE Trans. Autom. Control*, 32(5):432–435, 1987.

[10] E. C. Kerrigan. *Robust Constraint Satisfaction: Invariant Sets and Predictive Control*. PhD thesis, University of Cambridge, 2000.

[11] M. Schulze Darup and M. Mönnigmann. Accurate approximation of the largest null-controllable set for single-input bilinear systems. In *Proc. of 52th Conference on Decision and Control*, pp. 3951–3956, Florence, Italy, 2013.

[12] M. Schulze Darup and M. Mönnigmann. A stabilizing control scheme for linear systems on controlled invariant sets. *accepted for System and Control Letters*, 79:8–14, 2015.