Exercise 2

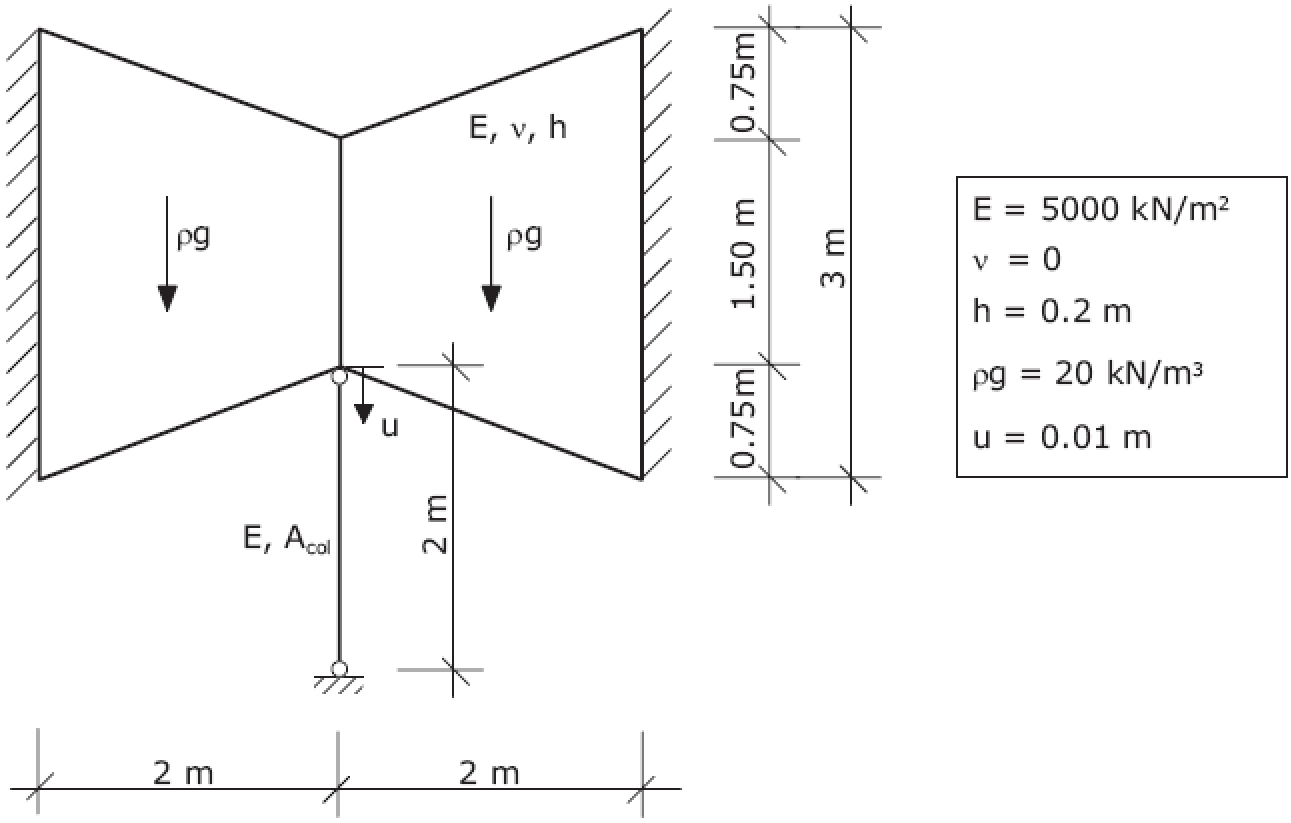



**Figure 1**: System and loading

The system shown in Figure 1 is loaded by the dead load of the plane slab elements. The cross section of the column $A_{col}$ has to be calculated such, that the vertical displacement is $u = 0.01\,m$.
Discretize the system using 4-node plane stress elements for the slabs and a 2-node truss element for the column.

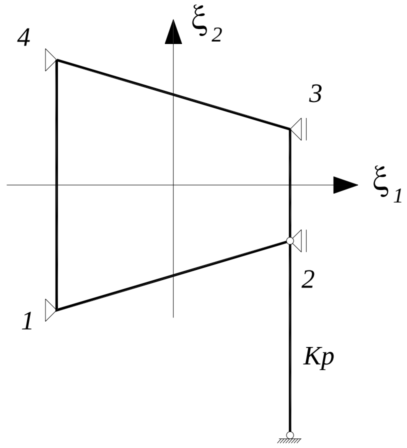
a) Determine all components of the vector of external loads and of the stiffness matrix which are necessary for the calculation of the required cross section area of the column and the unknown nodal displacements! For the entries of the stiffness matrix, a 1-point GAUSS-integration and for the entries of the load vector, a 2x2-point GAUSS-integration should be used.

b) Calculate the required cross section of the column and the unknown nodal displacement!

a) Physical coordinates of nodes:
b)

| node | $X_1^{ei}$[m] | $X_2^{ei}$[m] |
|---|---|---|
| 1 | 0.0 | 0.0 |
| 2 | 2.0 | 0.75 |
| 3 | 2.0 | 2.25 |
| 4 | 0.0 | 3.0 |

Using symetry :




Interpolation functions for bilinear quadrilateral element:

$$N^1(\boldsymbol{\xi}) = \frac{1}{4}(1-\xi_1)(1-\xi_2)$$
$$N^2(\boldsymbol{\xi}) = \frac{1}{4}(1+\xi_1)(1-\xi_2)$$
$$N^3(\boldsymbol{\xi}) = \frac{1}{4}(1+\xi_1)(1+\xi_2)$$
$$N^4(\boldsymbol{\xi}) = \frac{1}{4}(1-\xi_1)(1+\xi_2)$$

Values of interpolation functions and derivatives of shape function for $\xi_1^1 = \xi_2^1 = 0$

$$N^1(\boldsymbol{\xi}) = \frac{1}{4} \quad N^2(\boldsymbol{\xi}) = \frac{1}{4} \quad N^3(\boldsymbol{\xi}) = \frac{1}{4} \quad N^4(\boldsymbol{\xi}) = \frac{1}{4}$$
$$N^1_{,1}(\boldsymbol{\xi}) = -\frac{1}{4} \quad N^1_{,2}(\boldsymbol{\xi}) = -\frac{1}{4}$$
$$N^2_{,1}(\boldsymbol{\xi}) = \frac{1}{4} \quad N^2_{,2}(\boldsymbol{\xi}) = -\frac{1}{4}$$
$$N^3_{,1}(\boldsymbol{\xi}) = \frac{1}{4} \quad N^3_{,2}(\boldsymbol{\xi}) = \frac{1}{4}$$
$$N^4_{,1}(\boldsymbol{\xi}) = -\frac{1}{4} \quad N^4_{,2}(\boldsymbol{\xi}) = \frac{1}{4}$$

$$\mathbf{J} = \begin{bmatrix} \frac{\partial X_1}{\partial \xi_1} & \frac{\partial X_2}{\partial \xi_1} \\ \frac{\partial X_1}{\partial \xi_2} & \frac{\partial X_2}{\partial \xi_2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 0.0 & 0.0 \\ 2.0 & 0.75 \\ 2.0 & 2.25 \\ 0.0 & 3.0 \end{bmatrix} = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.125 \end{bmatrix}$$

$$\det(\mathbf{J}) = 1.125$$

$$\mathbf{J}^{-1} = \begin{bmatrix} \frac{\partial \xi_1}{\partial X_1} & \frac{\partial \xi_2}{\partial X_1} \\ \frac{\partial \xi_1}{\partial X_2} & \frac{\partial \xi_2}{\partial X_2} \end{bmatrix} = \frac{1}{\det(\mathbf{J})} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} = \frac{1}{1.125} \begin{bmatrix} 1.125 & 0.0 \\ 0.0 & 1.0 \end{bmatrix} = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 0.8889 \end{bmatrix}$$

$$\mathbf{B}_i(\xi) = \begin{bmatrix} \frac{\partial \xi_1}{\partial X_1} N^i_{,1}(\boldsymbol{\xi}) + \frac{\partial \xi_2}{\partial X_1} N^i_{,2}(\boldsymbol{\xi}) & 0 \\ 0 & \frac{\partial \xi_1}{\partial X_2} N^i_{,1}(\boldsymbol{\xi}) + \frac{\partial \xi_2}{\partial X_2} N^i_{,2}(\boldsymbol{\xi}) \\ \frac{\partial \xi_1}{\partial X_2} N^i_{,1}(\boldsymbol{\xi}) + \frac{\partial \xi_2}{\partial X_2} N^i_{,2}(\boldsymbol{\xi}) & \frac{\partial \xi_1}{\partial X_1} N^i_{,1}(\boldsymbol{\xi}) + \frac{\partial \xi_2}{\partial X_1} N^i_{,2}(\boldsymbol{\xi}) \end{bmatrix}$$

Node 2:

$$\mathbf{B}_2 = \begin{bmatrix} 0 \\ 0 + 0.8889 \cdot \left(-\frac{1}{4}\right) \\ 1.0 \cdot \frac{1}{4} + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.222 \\ 0.25 \end{bmatrix}$$

Node 3:

$$\mathbf{B}_3 = \begin{bmatrix} 0 \\ 0 + 0.8889 \cdot \frac{1}{4} \\ 1.0 \cdot \frac{1}{4} + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.222 \\ 0.25 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 \\ -0.222 & 0.222 \\ 0.25 & 0.25 \end{bmatrix}$$

$$\mathbb{C} = 5000\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0.5\end{bmatrix}$$

$$\mathbf{K}^{e1} = 2\cdot 2\cdot 0.2\cdot 1.125\cdot 5000\cdot\begin{bmatrix}0 & -0.222 & 0.25\\ 0 & 0.222 & 0.25\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0.5\end{bmatrix}\begin{bmatrix}0 & 0\\ -0.222 & 0.222\\ 0.25 & 0.25\end{bmatrix} = \begin{bmatrix}362.403 & -81.153\\ -81.153 & 362.403\end{bmatrix}$$

Load vector

$$\mathbf{r}^{e1} = \int_{\Omega^e} h\,\mathbf{N}^T\mathbf{b}\,d\Omega^e = h\int_{-1}^{1}\int_{-1}^{1}\mathbf{N}^T\mathbf{b}\,det(\mathbf{J})\,d\xi_1 d\xi_2 \approx h\sum_{i=1}^{2}\sum_{j=1}^{2}\mathbf{N}^T\mathbf{b}\,det(\mathbf{J})$$

$$N^2 = \frac{1}{4}(1+\xi_1)(1-\xi_2)$$

$$N^3 = \frac{1}{4}(1+\xi_1)(1+\xi_2)$$

Values of derivatives of shape functions for $\xi_1^1 = -1/\sqrt{3}\quad \xi_2^1 = -1/\sqrt{3}$

$$\begin{bmatrix}\frac{\partial N^1}{\partial\xi_1} & \frac{\partial N^2}{\partial\xi_1} & \frac{\partial N^3}{\partial\xi_1} & \frac{\partial N^4}{\partial\xi_1}\\ \frac{\partial N^1}{\partial\xi_2} & \frac{\partial N^2}{\partial\xi_2} & \frac{\partial N^3}{\partial\xi_2} & \frac{\partial N^4}{\partial\xi_2}\end{bmatrix} = \begin{bmatrix}-0.394 & 0.394 & 0.106 & -0.106\\ -0.394 & -0.106 & 0.106 & 0.394\end{bmatrix}$$

$$\mathbf{J} = \begin{bmatrix}\frac{\partial X_1}{\partial\xi_1} & \frac{\partial X_2}{\partial\xi_1}\\ \frac{\partial X_1}{\partial\xi_2} & \frac{\partial X_2}{\partial\xi_2}\end{bmatrix} = \begin{bmatrix}-0.394 & 0.394 & 0.106 & -0.106\\ -0.394 & -0.106 & 0.106 & 0.394\end{bmatrix}\begin{bmatrix}0.0 & 0.0\\ 2.0 & 0.75\\ 2.0 & 2.25\\ 0.0 & 3.0\end{bmatrix} = \begin{bmatrix}1.0 & 0.217\\ 0.0 & 1.342\end{bmatrix}$$

$\det(\mathbf{J}) = 1.342$

Values of derivatives of shape functions for $\xi_1^1 = 1/\sqrt{3}\quad \xi_2^1 = -1/\sqrt{3}$

$$\begin{bmatrix}\frac{\partial N^1}{\partial\xi_1} & \frac{\partial N^2}{\partial\xi_1} & \frac{\partial N^3}{\partial\xi_1} & \frac{\partial N^4}{\partial\xi_1}\\ \frac{\partial N^1}{\partial\xi_2} & \frac{\partial N^2}{\partial\xi_2} & \frac{\partial N^3}{\partial\xi_2} & \frac{\partial N^4}{\partial\xi_2}\end{bmatrix} = \begin{bmatrix}-0.394 & 0.394 & 0.106 & -0.106\\ -0.1057 & -0.394 & 0.394 & 0.106\end{bmatrix}$$

$$\mathbf{J} = \begin{bmatrix}\frac{\partial X_1}{\partial\xi_1} & \frac{\partial X_2}{\partial\xi_1}\\ \frac{\partial X_1}{\partial\xi_2} & \frac{\partial X_2}{\partial\xi_2}\end{bmatrix} = \begin{bmatrix}-0.394 & 0.394 & 0.106 & -0.106\\ -0.1057 & -0.394 & 0.394 & 0.106\end{bmatrix}\begin{bmatrix}0.0 & 0.0\\ 2.0 & 0.75\\ 2.0 & 2.25\\ 0.0 & 3.0\end{bmatrix} = \begin{bmatrix}1.0 & 0.2165\\ 0.0 & 0.9085\end{bmatrix}$$

$\det(\mathbf{J}) = 0.9085$

$$\mathbf{r}^{e1} = \int_{\Omega^e} h\,\mathbf{N}^T\mathbf{b}\,d\Omega^e = h\int_{-1}^{1}\int_{-1}^{1}\mathbf{N}^T\mathbf{b}\,det(\mathbf{J})\,d\xi_1 d\xi_2 \approx h\sum_{i=1}^{2}\sum_{j=1}^{2}\mathbf{N}^T\mathbf{b}\,det(\mathbf{J})$$

$$\mathbf{N} = \begin{bmatrix}0 & 0\\ N^2 & N^3\end{bmatrix} = \begin{bmatrix}0 & 0\\ \frac{1}{4}(1+\xi_1)(1-\xi_2) & \frac{1}{4}(1+\xi_1)(1+\xi_2)\end{bmatrix}$$

$$\mathbf{r}^{e1} = h\int_{-1}^{1}\int_{-1}^{1}\begin{bmatrix}0 & \frac{1}{4}(1+\xi_1)(1-\xi_2)\\ 0 & \frac{1}{4}(1+\xi_1)(1+\xi_2)\end{bmatrix}\begin{bmatrix}0\\ -\varrho g\end{bmatrix}det(\mathbf{J})\,d\xi_1 d\xi_2 = h\int_{-1}^{1}\int_{-1}^{1}\begin{bmatrix}-\frac{1}{4}(1+\xi_1)(1-\xi_2)\cdot\varrho g\\ -\frac{1}{4}(1+\xi_1)(1+\xi_2)\cdot\varrho g\end{bmatrix}det(\mathbf{J})\,d\xi_1 d\xi_2$$

$$\mathbf{r}^{e1} = -0.2\left(\begin{bmatrix}0.1667\cdot 20\\0.0447\cdot 20\end{bmatrix}\cdot 1.342 + \begin{bmatrix}0.6220\cdot 20\\0.1667\cdot 20\end{bmatrix}\cdot 0.9085 + \begin{bmatrix}0.1667\cdot 20\\0.6220\cdot 20\end{bmatrix}\cdot 0.9085 + \begin{bmatrix}0.0447\cdot 20\\0.1667\cdot 20\end{bmatrix}\cdot 1.342\right)$$

$$\mathbf{r}^{e1} = -0.2\left(\begin{bmatrix}0.1667+0.0447\\0.0447+0.1667\end{bmatrix}\cdot 20\cdot 1.342 + \begin{bmatrix}0.6220+0.1667\\0.1667+0.6220\end{bmatrix}\cdot 20\cdot 0.9085\right) = -0.2\left(\begin{bmatrix}5.674\\5.674\end{bmatrix}+\begin{bmatrix}14.33\\14.33\end{bmatrix}\right) = \begin{bmatrix}-4.0\\-4.0\end{bmatrix}$$

**Check:**

Calcultion of the other two nodal forces

$$\mathbf{N} = \begin{bmatrix}0 & 0\\N^1 & N^4\end{bmatrix} = \begin{bmatrix}0 & 0\\\frac{1}{4}(1-\xi_1)(1-\xi_2) & \frac{1}{4}(1-\xi_1)(1+\xi_2)\end{bmatrix}$$

$$\mathbf{r}^{e1} = h\int_{-1}^{1}\int_{-1}^{1}\begin{bmatrix}0 & \frac{1}{4}(1-\xi_1)(1-\xi_2)\\0 & \frac{1}{4}(1-\xi_1)(1+\xi_2)\end{bmatrix}\begin{bmatrix}0\\-\varrho g\end{bmatrix} det(\mathbf{J})d\xi_1 d\xi_2 = h\int_{-1}^{1}\int_{-1}^{1}\begin{bmatrix}-\frac{1}{4}(1-\xi_1)(1-\xi_2)\cdot\varrho g\\-\frac{1}{4}(1-\xi_1)(1+\xi_2)\cdot\varrho g\end{bmatrix} det(\mathbf{J})d\xi_1 d\xi_2$$

$$\mathbf{r}^{e1} = -0.2\left(\begin{bmatrix}0.6220+0.1667\\0.1667+0.6220\end{bmatrix}\cdot 20\cdot 1.342 + \begin{bmatrix}0.1667+0.0447\\0.0447+0.1667\end{bmatrix}\cdot 20\cdot 0.9085\right) = -0.2\left(\begin{bmatrix}21.169\\21.169\end{bmatrix}+\begin{bmatrix}3.841\\3.841\end{bmatrix}\right) = \begin{bmatrix}-5.0\\-5.0\end{bmatrix}$$

Weight of the element:

$$\mathbf{r}^{e1} = -\frac{3+1.5}{2}\cdot 2\cdot 0.2\cdot 20 = -18.0$$

**Solution of system of equations**

$$\begin{bmatrix}362.403+Kp & -81.153\\-81.153 & 362.403\end{bmatrix}\begin{bmatrix}u_2^2\\u_2^3\end{bmatrix} = \begin{bmatrix}-4.0\\-4.0\end{bmatrix}$$

$$(362.403+Kp)u_2^2 - 81.153\cdot u_2^3 = -4.0$$
$$-81.153\cdot u_2^2 + 362.403\cdot u_2^3 = -4.0$$

For $u_2^2 = -0.01$ we get:

$$u_2^3 = \frac{-4.0+81.153\cdot(-0.01)}{362.403} = -0.013277$$

$$Kp = \frac{-4.0+81.153(-0.013277)}{-0.01} - 362.403 = 145.344$$

**Check**

$$\begin{bmatrix}362.403+145.344 & -81.153\\-81.153 & 362.403\end{bmatrix}\begin{bmatrix}u_2^2\\u_2^3\end{bmatrix} = \begin{bmatrix}-4.0\\-4.0\end{bmatrix}$$

$$\begin{bmatrix}u_2^2\\u_2^3\end{bmatrix} = \begin{bmatrix}-0.0100\\-0.0133\end{bmatrix}$$

$$K_{col} = 2\cdot Kp,\qquad K_{col} = \frac{EA}{L}\ \Rightarrow A = \frac{L\cdot 2\cdot Kp}{E} = \frac{2.0\cdot 2\cdot 145.344}{5000} = 0.116\ m^2$$

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